Physics 605: Electrostatics

Due: never

Maxwell's equations are

$$\begin{array}{cc} \nabla \cdot \mathbf{E} = 4\pi\rho & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{array} \tag{1}$$

in *cgs* units. We also have the Lorenz force law,

$$\mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{2}$$

for a charge q moving with velocity v in an electric and magnetic field.

In the limit of electrostatics and magnetostatics, we assume that ρ, $\mathbf{E}$ and $\mathbf{B}$ do not change with time. In that case, we obtain

$$\begin{array}{cc} \nabla \cdot \mathbf{E} = 4\pi\rho & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = 0 & \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \end{array} \tag{3}$$

Our job, in electrostatics, is to solve these equations for $\mathbf{E}$ and $\mathbf{B}$, given ρ and $\mathbf{J}$.

1 Helmholtz theorem

Since Maxwell's equations are written in terms of the divergence and curl of a vector field, we could ask how much we know about a vector field if we specify just its divergence and curl. The answer is in the form of the Helmholtz theorem.

Theorem (Helmholtz): any vector field, $\mathbf{E}$, for which $|\mathbf{E}|$ vanishes more rapidly than $1/|\mathbf{r}|$ as $|\mathbf{r}| \to \infty$ can be written as $\mathbf{E} = -\nabla\phi + \nabla \times \mathbf{A}$, where

$$\begin{aligned} \phi &= -\frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{\nabla' \cdot \mathbf{E}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ \mathbf{A} &= \frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{\nabla' \times \mathbf{E}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \end{aligned}$$

That is, any vector field is the gradient of a scalar and the curl of another vector field.

To prove this, we assume $\mathbf{E} = -\nabla\phi + \nabla \times \mathbf{A}$. Computing the curl and divergence of this equation gives

$$\nabla \cdot \mathbf{E} = -\nabla^2\phi \qquad \nabla \times \mathbf{E} = \nabla^2\mathbf{A} - \nabla(\nabla \cdot \mathbf{A}). \tag{4}$$

The first equation is the Poisson equation, which we already know how to solve when $\mathbf{E}$ (and ϕ) falls of sufficiently rapidly. It's solution is

$$\phi = -\frac{1}{4\pi}\int d^3\mathbf{r}' \frac{\nabla'\cdot\mathbf{E}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{5}$$

The second equation seems more difficult. If we assume $\nabla\cdot\mathbf{A}=0$, however, we again obtain $\nabla\times\mathbf{E}=\nabla^2\mathbf{A}$ which we can solve for each individual component of $\mathbf{A}$. Consequently,

$$\mathbf{A} = \frac{1}{4\pi}\int d^3\mathbf{r}' \frac{\nabla'\times\mathbf{E}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

As a final step, let's try to compute $\nabla\cdot\mathbf{A}$ for this solution. In order to compute it, we need an identify:

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{7}$$

Therefore,

$$\begin{aligned}\nabla\cdot\mathbf{A} &= -\frac{1}{4\pi}\int d^3r'\,[\nabla'\times\mathbf{E}(\mathbf{r}')]\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{4\pi}\int d^3r'\nabla'\cdot\nabla'\times\mathbf{E}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned} \tag{8}$$

The last step is an integration-by-parts. Since the divergence of a curl is zero, we find that our solution naturally satisfies our constraint.

We might also ask whether we are missing something in this decomposition. In other words, maybe $\mathbf{E}=-\nabla\phi+\nabla\times\mathbf{A}+\mathbf{V}$. If so, however, $\nabla\cdot\mathbf{V}=0$ and $\nabla\times\mathbf{V}=0$. However, $\nabla\times\nabla\times\mathbf{V}=\nabla^2\mathbf{V}-\nabla(\nabla\cdot\mathbf{V})=\nabla^2\mathbf{V}=0$. So each component of V would necessarily satisfy Laplace's equation with $|\mathbf{V}|\to 0$ as $|\mathbf{r}|\to\infty$. As we will show in the next section, this implies that $\mathbf{V}=0$.

1.1 Uniqueness of solutions to Laplace's equation

Suppose $\nabla^2\phi=0$ and ϕ is specified on the boundary of some domain, $\mathcal{D}$. If $\nabla^2\phi_2=0$ and satisfies the same boundary condition then $\nabla^2(\phi-\phi_2)=0$ with $\phi-\phi_2=0$ on $\partial\mathcal{D}$. So here we have the same problem as in the last section.
Theorem: If $\nabla^2\phi=0$ and $\phi=0$ on the boundary of some domain, $\mathcal{D}$, then $\phi=0$ everywhere inside the domain.

> To prove this, consider $\int_{\mathcal{D}} d^3\mathbf{r}\phi\nabla^2\phi$, which is just zero. Integrating by parts gives
>
> $$0 = -\int_{\mathcal{D}} d^3\mathbf{r}\,(\nabla\phi)^2 + \oint_{\partial\mathcal{D}} d\mathbf{a}\cdot\nabla\phi\;\phi. \tag{9}$$

The boundary term, obviously, equals zero so $\int_{\mathcal{D}} d^3\mathbf{r}\,(\nabla\phi)^2 = 0$. Therefore, ϕ is a constant and must, in fact, equal 0.

If we example the proof again, we see that it still works if we specify $\mathbf{n}\cdot\nabla\phi$, where $\mathbf{n}$ is the unit normal vector to $\partial\mathbf{D}$.

1.2 Curl-free vector fields

The electric field has an interesting and special property, $\nabla\times\mathbf{E} = 0$. Now we see, from the Helmholtz theorem, that $\mathbf{E} = -\nabla\phi$.

Suppose we take a small "disk" in space, called D. This doesn't have to be a perfect disk; it need only have the topology of a disk. This means, in particular, that it is two-dimensional and that its boundary, denoted ∂D, is a closed curve in space. Stokes' theorem then tells us that

$$\int_D d\mathbf{a}\cdot\nabla\times\mathbf{E} = \oint_{\partial D} d\mathbf{l}\cdot\mathbf{E} = 0. \tag{10}$$

This is independent of the shape of the disk, D, itself.

There is a lot to decompress in Stokes' theorem mathematically. First, we could ask how we compute the two integrals involved. Second, we could ask why Stokes' theorem is true at all. I refer you to Griffitth's excellent book on electricity and magnetism on the why of Stokes' theorem (or for those of you crazy enough, Spivak's book "Calculus on Manifolds").

Let's try to figure out how to compute the two integrals. Let's start with the right-most integral – the path integral. Let's define a path, $\mathbf{l}(s)$, not necessarily closed, as a function of a single parameter. Then $d\mathbf{l} = ds\; d\mathbf{l}/ds$. The vector

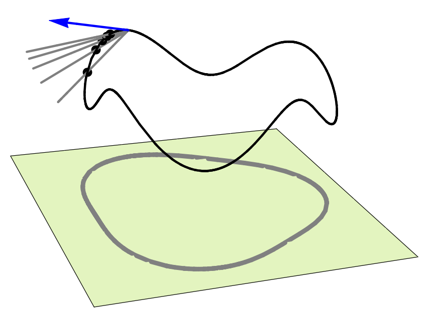

Figure 1: The infinitesimal element $d\mathbf{l}(s)$ defined as a limiting process.

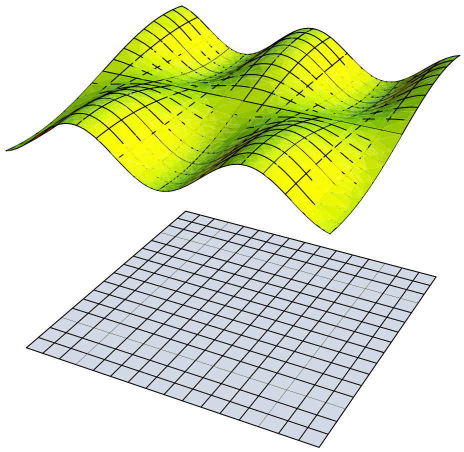

Figure 2: A surface is decomposable into infinitesimal parallelograms whose area is given by the cross-product of the vectors spanning two adjacent sides.

$d\mathbf{l}/ds$ is always tangent to $\mathbf{l}(s)$. To see this, consider the limiting process shown in Fig. 1. Hence,

$$\int_{s_1}^{s_2} ds \; \frac{d\mathbf{l}}{ds} \cdot \mathbf{E}\left(\mathbf{l}(s)\right). \tag{11}$$

Since $\mathbf{E} = -\nabla\phi$, we have

$$\begin{aligned} \int_{s_1}^{s_2} ds \; \frac{d\mathbf{l}}{ds} \cdot \mathbf{E}\left(\mathbf{l}(s)\right). &= \int_{s_1}^{s_2} ds \; \frac{d\mathbf{l}}{ds} \cdot \nabla\phi\left(\mathbf{l}(s)\right) \\ &= \int_{s_1}^{s_2} ds \; \frac{d}{ds}\phi\left(\mathbf{l}(s)\right) = \phi(\mathbf{l}(s_2)) - \phi(\mathbf{l}(s_1)). \end{aligned} \tag{12}$$

Hence, we see directly that $\oint d\mathbf{l} \cdot \mathbf{E} = 0$ for a curl-free vector field.

The left-hand side is more complicated to compute. To do so, we must write an equation for a surface, $\mathbf{R}(\xi_1, \xi_2)$. We can use this parametrization to write down two vectors that are tangent to the surface, $\partial_1\mathbf{R}$ and $\partial_2\mathbf{R}$ (think about why this must be so). Then

$$d\mathbf{a} = d\xi_1 d\xi_2 \; \partial_1\mathbf{R} \times \partial_2\mathbf{R}. \tag{13}$$

Why? Consider decomposing the surface into infinitesimal paralellograms as in Fig. 2. The infinitesimal parallelogram spanned by $d\xi_1\partial_1\mathbf{R}$ and $d\xi_2\partial_2\mathbf{R}$. The magnitude of the cross-product is simply the area of the parallelogram. The direction is the unit normal vector to the parallelogram. Hence,

$$\int_D d\mathbf{a} \cdot \nabla \times \mathbf{E} = \int d\xi_1 d\xi_2 \; \partial_1 \mathbf{R} \times \partial_2 \mathbf{R} \cdot (\nabla \times \mathbf{E}) \,. \tag{14}$$

2 Gauss' Law

Theorem (Gauss?): Let V be a volume with boundary ∂V. Then

$$\int_V d^3x \; 4\pi\rho = \int_V d^3x \; \nabla \cdot \mathbf{E} = \int_{\partial V} d\mathbf{a} \cdot \mathbf{E}, \tag{15}$$

where $d\mathbf{a}$ is the infinitesimal area element on ∂V and is oriented outward with respect to the volume, V.

We can use this theorem to find some important, particularly symmetric solutions. First, consider a sphere of radius R with a uniform charge density, ρ. What is the electric field outside of this sphere? Since the charge distribution is symmetric with respect to rotations, the electric field must also be. In addition, since the charge distribution is invariant under reflections, $\mathbf{E}$ must also be. These symmetries are highly constraining. In particular, they require that $\mathbf{E} \propto E(r)\hat{\mathbf{r}}$ so that the electric field is always pointing radially outward or inward.

Consider a spherical region of radius $r > R$. Then $\int_V d^3x \; 4\pi\rho = 4\pi Q$, where Q is the total charge. However,

$$\int_{\partial V} d\mathbf{a} \cdot \mathbf{E} = 4\pi r^2 \; E(r). \tag{16}$$

Hence, $E(r) = Q/r^2$ and $\mathbf{E} = Q\hat{\mathbf{r}}/r^2$.

Let's consider the limit that $R \to 0$ while $\int_R d^3x \; \rho = Q$ remains constant. This is the limit of a point charge. Clearly, we conclude, the electric field of a point charge is

$$\mathbf{E}(r) = \frac{Q\mathbf{r}}{r^2}. \tag{17}$$

There is one difficulty here. Suppose you were to calculate $\nabla \cdot \mathbf{E}$ – you would find 0. Then, by Gauss' law,

$$\int d^3x \; \nabla \cdot \mathbf{E} = \int d^3x \; 0 = 4\pi Q. \tag{18}$$

We also know that $\rho = 0$ for any point away from the origin. What we conclude is that $\nabla \cdot \mathbf{E} = \delta(x)\delta(y)\delta(z)$. So we have the following result,

$$-\nabla^2 \left(\frac{Q\hat{\mathbf{r}}}{r^2} \right) = 4\pi\delta^3(\mathbf{x}), \tag{19}$$

our first solution to the Poisson equation.

3 Boundary conditions

3.1 Charged surfaces

Often in electrostatics, one has a charged surface, with charge per unit area σ. Consider a small cylinder of radius r and height h perpendicular through the surface. The total charge enclosed is $Q = \sigma\pi r^2$. Gauss' law tells us that

$$\oint d\mathbf{a} \cdot \mathbf{E} = 4\pi Q. \tag{20}$$

This integral decomposes into three distinct parts, the two end caps of the cylinder and the actual round part of the cylinder. Notice that,

$$\int_{\text{round part}} d\mathbf{a} \cdot \mathbf{E} \to 0 \tag{21}$$

as $r \to 0$ and $h \to 0$. Therefore, only the end caps contribute to this integral. Hence,

$$\pi r^2 \left[\mathbf{n} \cdot \mathbf{E}|_+ - \mathbf{n} \cdot \mathbf{E}|_- \right] \approx 4\pi\sigma\pi r^2, \tag{22}$$

where $\mathbf{n}$ is normal to the surface and the $\pm$ refer to the electric fields on either side of the surface, with $+$ in the direction of the surface normal. Hence, $\mathbf{E}$ is discontinuous across a charged surface with

$$\mathbf{n} \cdot \mathbf{E}|_+ - \mathbf{n} \cdot \mathbf{E}|_- = 4\pi\sigma. \tag{23}$$

Therefore,

$$\mathbf{n} \cdot \nabla\phi|_+ - \mathbf{n} \cdot \nabla\phi|_- = -4\pi\sigma. \tag{24}$$

This tells us what happens to the normal component of $\mathbf{E}$. What about the tangential components? To answer this, consider a very small rectangular loop penetrating the surface oriented so one side is normal to the surface. Let's call $\mathcal{A}$ the area contained by the loop and $\partial\mathcal{A}$ the rectangular boundary. Then Stokes' theorem says

$$\oint_{\partial\mathcal{A}} d\mathbf{l} \cdot \mathbf{E} = \int_{\mathcal{A}} d\mathbf{a} \cdot \nabla \times \mathbf{E} = 0. \tag{25}$$

The left hand side of Eq. (25) can be decomposed into its individual sides. These give

$$\sum_{i=1}^{4} \int_i d\mathbf{l} \cdot \mathbf{E} = 0, \tag{26}$$

where the subscript i labels which of the four sides we are considering. In the limit that the area of the loop vanishes, we find that every pair of adjacent sides cancel. In particular, $\mathbf{t} \cdot \mathbf{E}|_+ - \mathbf{t} \cdot \mathbf{E}|_- = 0$ for any vector tangent to the surface, $\mathbf{t}$. In terms of the scalar potential, ϕ, this gives

$$\mathbf{t} \cdot \nabla\phi|_+ = \mathbf{t} \cdot \nabla\phi|_- . \tag{27}$$

Suppose that ϕ is discontinuous across the charged boundary. Then there is no reason for the tangential derivatives of ϕ to be continuous. Similarly, ϕ cannot be continuous if the tangential derivatives themselves are discontinuous. This boundary condition, therefore, translates to the need to keep ϕ continuous, even as its normal derivative is not.

3.2 Conducting surfaces

If a material is a conductor, the charges are mobile. Since like charges repel, we know that all the excess charge in a conductor will migrate to the surface. And the charges themselves must arrange themselves on that surface so that the electric field is zero. Why? Because if it was not zero, the charges would move and, therefore, not be in equilibrium. Finally, we conclude that the potential ϕ at a conductors surface is constant. And any conductor which is connected to another conductor must be at the same constant potential.

Even at constant potential, this does not mean that a conductor has no excess charge. Indeed, the boundary conditions for the electric field and the vanishing of the electric field inside the conductor immediately imply,

$$\mathbf{n} \cdot \mathbf{E}|_{\text{boundary}} = 4\pi\sigma. \tag{28}$$

3.3 Dipole surfaces

There is another kind of surface to consider – a surface of dipoles. The best approach to this is to imagine you have a slab of material of width W with a negative surface charge density σ on one boundary and $-\sigma$ on the other boundary. For now, assume the slab is flat and perpendicular to the $\hat{\mathbf{z}}$ axis. One boundary is at $z = -W/2$ and the other at $z = W/2$.

How do we obtain a surface of dipoles from this? Consider a patch of charge, of area dA, on the surface, $\pm dQ = \pm\sigma dA$. A pure dipole is a configuration of charge that has only terms with $\ell = 1$ and no other. On a homework, you proved that two opposite charges, $\pm q$, separated by a become a pure dipole in the limit that $q \to 0$ while qa remains constant. We obtain pure dipoles by taking the limit $W \to 0$ while dQW remains constant.

There are three regions in this problem, two away from the slab and one inside the slab. Therefore,

$$\phi = \begin{cases} a_- + b_-(z + W/2), & z < -W/2 \\ a_0 + b_0 z, & -W/2 < z < W/2 \\ a_+ + b_+(z - W/2), & z > W/2. \end{cases} \tag{29}$$

The continuity of ϕ implies that $a_- = a_0 - b_0 W/2$ and $a_+ = a_0 + b_0 W/2$. The discontinuity of $\partial_z \phi$ tells us that $b_0 - b_- = -4\pi\sigma$ and $b_+ - b_0 = 4\pi\sigma$. Putting these together gives a potential

$$\phi = \begin{cases} a_0 - b_0 W/2 + (b_0 + 4\pi\sigma)(z + W/2), & z < -W/2 \\ a_0 + b_0 z, & -W/2 < z < W/2 \\ a_0 + b_0 W/2 + (b_0 + 4\pi\sigma)(z - W/2), & z > W/2. \end{cases} \tag{30}$$

Now consider $\partial_z \phi(z = -W/2) = b_0 + 4\pi\sigma$. This is the magnitude of the electric field on one side of the slab, $E|_-$. Therefore, $b_0 = E|_- - 4\pi\sigma$. On the other side of the slab, $\partial_z \phi(z = W/2) = b_0 + 4\pi\sigma$, so apparently the electric field is continuous as $W \to 0$. However, if we compute $\phi(z = W/2) - \phi(z = -W/2) = b_0 W = E|_- W - 4\pi\sigma W$. Letting $\sigma W \equiv p$, the dipole moment per unit area, be constant, we obtain the startling result that

$$\phi(z = W/2) - \phi(z = -W/2) = -4\pi p. \tag{31}$$

In other words, the potential itself is discontinuous across a dipole surface, but not the electric field. Since surfaces are all locally flat and W is always a small length, this result holds even for curved surfaces.

4 An example problem

Consider a conducting sphere of radius R held at potential 0 in an applied electric field, $\mathbf{E}_0$. Find the electric field everywhere.

Without loss of generality, we assume $\mathbf{E}_0 = E_0 \hat{\mathbf{z}}$ so that the problem has azimuthal symmetry. Everywhere outside the sphere, the potential is

$$\phi = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} \frac{a_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{32}$$

We require $\phi = 0$ when $r = R$ so

$$0 = -E_0 R P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{a_\ell}{R^{\ell+1}} P_\ell(\cos\theta). \tag{33}$$

Since the P_ℓ are orthogonal, we see that only the a_1 coefficient is nonzero. And, in particular,

$$a_1 = E_0 R^3, \tag{34}$$

so

$$\phi = -E_0 \left[r + \frac{R^3}{r^2} \right] \cos\theta. \tag{35}$$

Now we can compute the surface charge density on the surface of the conducting sphere. This yields

$$\sigma = \frac{E_0}{2\pi} \cos\theta. \tag{36}$$

5 Method of images - repeated from Green functions

Let's consider the problem of solving

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = 4\pi\delta^3(\mathbf{r} - \mathbf{r}') \tag{37}$$

in the positive half space $z > 0$ with the boundary condition that $G = 0$ when $z = 0$. We also assume that $\phi(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$.

We're going to derive a solution to this problem by taking advantage of the one we already have. In order to make our solution work, however, we need to prove something about the Laplacian that hasn't quite come up before.

Theorem: A solution to Laplace's equation in a domain $\mathcal{D}$ for which $\phi(\mathbf{r})$ is specified on the boundary $\partial\mathcal{D}$ is unique.

> Suppose there were two solutions, ϕ_1 and ϕ_2, sharing the same boundary condition on $\partial\mathcal{D}$. Then $\phi = \phi_1 - \phi_2$ would also be a solution with $\phi = 0$ on $\partial\mathcal{D}$. What we really want to prove, then, is that $\phi = 0$ on $\partial\mathcal{D}$ implies it is zero throughout $\mathcal{D}$. If so, then $\phi_1 = \phi_2$ and we would have uniqueness.
>
> To prove this last part, consider $\nabla(\phi\nabla\phi) = (\nabla\phi)^2 + \phi\nabla^2\phi$. Since $\nabla^2\phi = 0$ by construction, we have $\nabla(\phi\nabla\phi) = (\nabla\phi)^2$. Integrating

both sides over volume and using the divergence theorem on the left side, we obtain

$$\oint_{\partial\mathcal{D}} d\mathbf{a}\cdot\nabla\phi\;\phi = \int_{\mathcal{D}} dV\;(\nabla\phi)^2. \tag{38}$$

The boundary condition we derived for ϕ implies that the left hand side is zero. Therefore, the right hand side must be as well. Since the integrand is always positive or zero, it must be zero. Therefore, we obtain the result that ϕ must be constant and, since it is zero on the boundary, it must be zero everywhere.

Uniqueness helps us because of the following fact:

$$\phi = \frac{1}{|\mathbf{r} - x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}|} - \frac{1}{|\mathbf{r} - x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} - z'\hat{\mathbf{z}}|} \tag{39}$$

vanishes on the $z = 0$ plane. Consequently, when $z > 0$, it solves the Poisson equation with precisely the boundary condition we want. Since the solution is unique, we can stop looking for solutions.

Indeed, if $G(x - x', y - y', z - z')$ is a Green function whose arguments are even, then $G(x - x', y - y', z - z') - G(x - x', y - y', z + z')$ always vanishes on the $z = 0$ plane. The method of images turns out to be useful in many different linear equations!

6 Energy

We can also associate an energy with an electrostatic configuration of charges, though we must be oddly clever to do this properly. Suppose you have a configuration of charges with electrostatic potential $\phi(\mathbf{r})$. Then the work of bringing a new charge, q, from infinity to a definite position $\mathbf{r}$ is space can be computed to be

$$W = q\int d\mathbf{l}\cdot\nabla\phi = q\phi(\mathbf{r}). \tag{40}$$

Therefore, the work required to bring N charges, $q_1, q_2, \cdots, q_N$ to positions $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$ is

$$\begin{aligned} W &= \sum_{i<j}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \\ &= \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \end{aligned} \tag{41}$$

The last step simply takes advantage of the symmetry to increase the range of the sum.

We want to take the expression for the work and rewrite it in terms of the electric field. This almost works – if we define $\phi_i(\mathbf{r})$ as the potential for all the charges but the i^{th}, then $W = (1/2)\sum_{i=1}^{N} q_i\phi_i(\mathbf{r}_i)$. Let's suppose instead, however, that we don't have Dirac delta point charges, but lumps of charge described by a distribution $\rho_0(\mathbf{r})$. The force on a lump of charge then generalizes to

$$\mathbf{F} = \int d^3r\rho_i(\mathbf{r}-\mathbf{r}_i)\phi(\mathbf{r}) = \int d^3r\rho_i(\mathbf{r})\phi(\mathbf{r}+\mathbf{r}_i). \tag{42}$$

Similarly, the work done moving a lump of charge into an already existing distribution of charged lumps becomes

$$\begin{aligned} W &= \frac{1}{2}\sum_{i\neq j}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_j(\mathbf{r}'-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{2}\sum_{ij}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_j(\mathbf{r}'-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{2}\sum_{i=1}^{N}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_j(\mathbf{r}'-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}'|}. \end{aligned}$$

Since $\sum_{i=1}^{N}\rho_i(\mathbf{r}-\mathbf{r}_i) = \rho(\mathbf{r})$ is the entire distribution of charge we have assemblied, we find

$$\begin{aligned} W &= \frac{1}{2}\int d^3rd^3r'\,\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{2}\sum_{i=1}^{N}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_i(\mathbf{r}'-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{2}\int d^3r\rho(\mathbf{r})\phi(\mathbf{r}) - \frac{1}{2}\sum_{i=1}^{N}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_i(\mathbf{r}'-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}'|} \\ &= -\frac{1}{8\pi}\int d^3r\nabla^2\phi\ \phi - \frac{1}{2}\sum_{i=1}^{N}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_i(\mathbf{r}'-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{8\pi}\int d^3r\mathbf{E}^2(\mathbf{r}) - \frac{1}{2}\sum_{i=1}^{N}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_i(\mathbf{r}'-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}'|}. \end{aligned} \tag{43}$$

The last term is actually independent of the positions $\mathbf{r}_i$. Indeed, it does not vanish when the particles are infinitely far apart; it represents the work required to assemble a single lump of charge at infinite – the self energy of a single charge. We write

$$E_{self} = \frac{1}{2}\sum_{i=1}^{N}\int d^3rd^3r'\,\frac{\rho_i(\mathbf{r}-\mathbf{r}_i)\rho_i(\mathbf{r}'-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}'|}. \tag{44}$$

Notice that this expression is actually independent of the $\mathbf{r}_i$, as we would expect. It is just some finite contribution to the energy of the field and, since the energy is ambiguous up to a constant. we can simply neglect to worry about it in our computations.

Then we assign the energy stored in an electric field as the work done to assemble the charges at their final positions *plus* the work required to assemble the charges. This gives us the strangely simple result

$$U = \int d^3r \frac{\mathbf{E}^2}{8\pi}. \tag{45}$$

When we work with continuous distributions of charge, this result will always be finite anyway. When we are working with Dirac delta functions (which are approximations to whatever a charge actually is), we must be careful not to compute the charge self-energy.

7 Dielectrics

A great deal of electrostatics is really about trying to understand electric fields in matter (or some other complex configuration). To proceed, we need to develop a model of matter and how it responds to electric fields.

7.1 Polarization

If we assume that an electric field induces small dipole moments, we can describe the dipole moment per volume, the polarization $\mathbf{P}$, with a vector field. The polarization is a function of the electric field, and in particular, we know that $\mathbf{P}(\mathbf{E}=0)=0$. If $\mathbf{E}$ is small enough, we can Taylor expand $\mathbf{P}$,

$$\mathbf{P} = \sum_{j=1}^{3} \chi ij \mathbf{E}_j + \mathcal{O}(\mathbf{E}^2). \tag{46}$$

The χij is the **dielectric susceptibility** and it depends on the material. For many materials, $\chi ij = \chi \delta_{ij}$.

The polarization itself affects the electric field. To understand how, we need to think about the electric field induced by a single dipole at a position $\mathbf{r}_0$. Consider a dipole expansion about the point $\mathbf{r}_0$. A pure dipole is a configuration of charge that has only terms with $\ell = 1$ and no other. On a homework, you proved that two opposite charges, $\pm q$, separated by a become a pure dipole in the limit that $q \to 0$ while qa remains constant. Therefore, a pure dipole has

$$\phi = \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \cdot \mathbf{p}, \tag{47}$$

where $\mathbf{p}$ is the dipole moment. For a polarization, the dipole moment per unit area is $\mathbf{P}(\mathbf{r}_0)d^3r_0$. Therefore,

$$\begin{aligned}
\phi &= \int d^3r_0 \mathbf{P}(\mathbf{r}_0) \cdot \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \\
&= -\int d^3r_0 \mathbf{P}(\mathbf{r}_0) \cdot \nabla_0 \frac{1}{|\mathbf{r}-\mathbf{r}_0|} \\
&= \int d^3r_0 \left[\nabla_0 \cdot \mathbf{P}(\mathbf{r}_0)\right] \frac{1}{|\mathbf{r}-\mathbf{r}_0|}.
\end{aligned} \tag{48}$$

This is the potential for the electric field in a charge distribution $\nabla \cdot \mathbf{P}$, so

$$\nabla \cdot \mathbf{E} = 4\pi \nabla \cdot \mathbf{P}(\mathbf{r}). \tag{49}$$

If there are additional charges in the system, we have $\nabla \cdot \mathbf{E} = 4\pi\sigma + 4\pi\nabla \cdot \mathbf{P}$.

Now we can combine this with $\mathbf{P} = \hat{\chi}\mathbf{E}$, where $\hat{\epsilon}$ is the dielectric tensor, to obtain

$$\nabla \cdot \left[\left(\hat{1} - 4\pi\hat{\chi}\right)\mathbf{E}\right] = 4\pi\sigma. \tag{50}$$

We usually call $\left(\hat{1} - 4\pi\hat{\chi}\right)\mathbf{E} = \mathbf{D}$. Finally, we call $\hat{\epsilon} = \hat{1} - 4\pi\hat{\chi}$ the dielectric tensor.

7.2 Boundary conditions

In a dielectric material, we find that $-\hat{\epsilon}\nabla\phi = \mathbf{D}$. At a boundary between two regions with different dielectric constants, ϵ_1 and ϵ_2, we can develop boundary conditions by noting that $\nabla \times \mathbf{E} = 0$ but $\nabla \cdot \mathbf{D} = 4\pi\rho$. The curl equation for $\mathbf{E}$ tells us that $\mathbf{t} \cdot \mathbf{E}$ is continuous across a boundary by the same arguments we used for charged surfaces. On the other hand, the divergence theorem says that $\mathbf{n} \cdot \mathbf{D}|_+ - \mathbf{n} \cdot \mathbf{D}|_- = 4\pi\sigma$, where σ is a surface charge density at the boundary between the two dielectric materials.

In terms of the electric potential, we therefore have that ϕ is continuous across a surface and

$$-\mathbf{n}^T\left(\hat{\epsilon}\right)\nabla\phi|_-^+ = 4\pi\sigma. \tag{51}$$

If a material is isotropic, $\hat{\epsilon}$ is diagonal and, therefore, this simplifies to

$$\epsilon\mathbf{n} \cdot \nabla\phi|_-^+ = -4\pi\sigma \tag{52}$$

7.3 A dielectric sphere

A dielectric sphere of radius R with an isotropic dielectric constant, $\hat{\epsilon} = \epsilon\hat{1}$ is placed in an external electric field $\mathbf{E} = E\hat{\mathbf{z}}$. We will proceed to find the electric potential everywhere. Taking advantage of azimuthal symmetry and keeping track of only the $\ell = 1$ terms, we have

$$\phi = \begin{cases} a_1 r \cos\theta & r < R \\ \frac{b_1}{r^2}\cos\theta + Er\cos\theta & r > R \end{cases} \tag{53}$$

The boundary conditions are

$$\begin{aligned} a_1 R - b_1/R^2 + ER &= 0 \\ -2b_1 R^{-3} - E + \epsilon a_1 &= 0. \end{aligned}$$

Therefore, $a_1R^3 + ER^3 = b_1$ and $\epsilon a_1 - E - 2R^{-3}(a_1R^3 + ER^3) = (\epsilon - 2)a_1 - 3E = 0$. Therefore,

$$\phi = \begin{cases} \frac{3E}{\epsilon - 2} r\cos\theta & r < R \\ \frac{ER^3}{r^2}\left(\frac{\epsilon+1}{\epsilon-2}\right)\cos\theta + Er\cos\theta & r > R \end{cases} \tag{54}$$

7.4 Method of images

The method of images also works for dielectric materials. Consider an isotropic dielectric ϵ in the half space $z < 0$. We are going to find the Green function associated with a charge in the space $z > 0$. Taking advantage of translational symmetry, we can place the charge at a position $(x, y, z) = (0, 0, a > 0)$. Then

$$G(\mathbf{r}, \mathbf{r}') = \begin{cases} \frac{1}{|\mathbf{r} - a\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r} - b\hat{\mathbf{z}}|}, & z > 0 \\ \frac{q'}{|\mathbf{r} - c\hat{\mathbf{z}}|}, & z < 0. \end{cases} \tag{55}$$

Now we need to choose b, c, q and q' to match the appropriate boundary conditions. These are

$$\frac{1}{\sqrt{r^2 + a^2}} + \frac{q}{\sqrt{r^2 + b^2}} = \frac{q'}{\sqrt{r^2 + c^2}} \tag{56}$$

$$\frac{a}{(r^2 + a^2)^{3/2}} + \frac{qb}{(r^2 + b^2)^{3/2}} = \frac{\epsilon q' c}{(r^2 + c^2)^{3/2}}. \tag{57}$$

These must be true for any r. Therefore, the denominators should all be equal in magnitude. Therefore, $a = -b = c$. Therefore,

$$1 + q = q' \tag{58}$$

$$a - qa = \epsilon q' a \tag{59}$$

or $1-q=\epsilon q'$ and $1+q=q'$. Therefore, $q'=2/(1+\epsilon)$ and $q=(1-\epsilon)/(1+\epsilon)$. Therefore,

$$G(\mathbf{r},\mathbf{r}')=\begin{cases}\frac{1}{|\mathbf{r}-a\hat{\mathbf{z}}|}+\frac{1-\epsilon}{1+\epsilon}\frac{1}{|\mathbf{r}+a\hat{\mathbf{z}}|}, & z>0\\ \frac{2}{1+\epsilon}\frac{1}{|\mathbf{r}-a\hat{\mathbf{z}}|}, & z<0.\end{cases} \tag{60}$$

A similar calculation can find $G(\mathbf{r},\mathbf{r}')$ when $\mathbf{r}'$ is in the dielectric.

7.5 Energy inside a dielectric

The energy of a dielectric must account for both the presence of a charge and the energy required to assemble the dipoles due to the polarization $\mathbf{P}$. Suppose we know the electric potential ϕ. Then adding a charge gives $\delta\rho$ gives

$$\delta U=\int d^3r\ \phi(\mathbf{r})\delta\rho(\mathbf{r})=\frac{1}{4\pi}\int d^3r\ \phi\nabla\cdot\delta\mathbf{D}=\frac{1}{4\pi}\int d^3r\ \mathbf{E}\cdot\delta\mathbf{D}. \tag{61}$$

Using $\mathbf{E}=\mathbf{D}/\epsilon$, we can integrate both sides. This gives

$$U=\frac{1}{8\pi}\int d^3r\ \mathbf{E}\cdot\mathbf{D}. \tag{62}$$

8 Capacitance

Consider two conducting surfaces of equal magnitude but opposite sign total charge Q. The electric potential difference between the two surfaces, V, must be linearly related to Q. Therefore, we write $Q=CV$, and call C the **capacitance**.

9 Other coordinate systems

We've taken for granted that we can write the Laplacian in cartesian, cylindrical and spherical coordinates. There are many other coordinate systems possible from which we can solve for Green functions. To establish these, we need to think about coordinate systems more generally. Consider a set of cartesian coordinates $\mathbf{r}$. A set of new coordinates $\vec{\xi}$ can be defined by an invertible function, $\vec{\xi}(\mathbf{r})$. We can define a set of basis vectors from this map by $\partial_i\vec{\xi}$ which give us lengths in the $\vec{\xi}$ space in terms of lengths in cartesian coordinates, $d\vec{\xi}=(d\mathbf{r}\cdot\nabla)\vec{\xi}$. Similarly, the inverse map $\mathbf{r}(\vec{\xi})$ implies $d\mathbf{r}=(d\vec{\xi}\cdot\nabla_\xi)\mathbf{r}$.

We'll specialize to orthogonal coordinates, where $\partial_{\xi_i}\mathbf{r}$ are mutually orthogonal. Define

$$\partial_{\xi_i}\mathbf{r}=\ell_i\hat{\mathbf{e}}_i, \tag{63}$$

where $\hat{\mathbf{e}}_i$ is a unit vector.

The dot product must also be modified. Note that, if $\mathbf{v} = \sum_i v_i \partial_{\xi_i} \mathbf{r}$, then

$$\mathbf{v} \cdot \mathbf{w} = \sum_{ij} v_i w_j \partial_{\xi_i} \mathbf{r} \cdot \partial_{\xi_j} \mathbf{r}. \tag{64}$$

For orthogonal coordinates,

$$\mathbf{v} \cdot \mathbf{w} = \sum_i v_i w_i \ell_i^2. \tag{65}$$

9.1 The volume integral

We can use this map between coordinate systems to define an infinitesimal volume element. The triple product between three vectors, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ gives the volume of a parallel-piped spanned by the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The volume element is, therefore,

$$dV = d^3\xi \ \partial_1 \mathbf{r} \cdot (\partial_2 \mathbf{r} \times \partial_3 \mathbf{r}) \,. \tag{66}$$

For orthogonal coordinates,

$$dV = d^3\xi \ell_1 \ell_2 \ell_3. \tag{67}$$

9.2 Gradient, divergence, and Laplacian

To compute the gradient of the function in the new coordinate system, we'll consider a path through space from a point $\vec{\xi}_{start}$ to $\vec{\xi}_{end}$ with two fixed coordinates. The fundamental theorem of vector calculus says

$$\int d\mathbf{l} \cdot \nabla \phi = \phi(\vec{\xi}_{end}) - \phi(\vec{\xi}_{start}). \tag{68}$$

Let's say the path lies along the ξ_i direction. Then we can rewrite the above equation to

$$\int d\xi_i \ell_i \hat{\mathbf{e}}_i \cdot \nabla \phi = \int_{\xi_{i,start}}^{\xi_{i,end}} d\xi_i \frac{\partial \phi}{\partial \xi_i}, \tag{69}$$

where the right integral is taken at fixed $\xi_{j \neq i}$. Since this doesn't depend on where the path is and how long it is, the integrands must themselves be equal. Consequently, we find that

$$\nabla \phi = \sum_{i=1}^{3} \frac{1}{\ell_i} \frac{\partial \phi}{\partial \xi_i} \hat{\mathbf{e}}_i. \tag{70}$$

To find the divergence, we can take advantage of

$$\int dV\ \nabla \cdot \mathbf{v} = \oint d\mathbf{a} \cdot \mathbf{v}. \tag{71}$$

In particular, choose a domain which is rectangular in the coordinates $\vec{\xi}$ spanning from a corner $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ to antipodal corner $(\bar{\xi}'_1, \bar{\xi}'_2, \bar{\xi}'_3)$. Note that $dV = \ell_1\ell_2\ell_3$. On the other hand,

$$d\mathbf{a} = d\xi_2 d\xi_3 \ell_2\ell_3 \hat{\mathbf{e}}_1\Big|_{(\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3)}^{(\bar{\xi}'_1,\bar{\xi}_2,\bar{\xi}_3)} + d\xi_3 d\xi_1 \ell_3\ell_1 \hat{\mathbf{e}}_2\Big|_{(\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3)}^{(\bar{\xi}_1,\bar{\xi}'_2,\bar{\xi}_3)} + d\xi_1 d\xi_2 \ell_1\ell_2 \hat{\mathbf{e}}_3\Big|_{(\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3)}^{(\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}'_3)}. \tag{72}$$

That is, the area integral decomposes into the individual areas of cube faces. Then we use the fundamental theorem of calculus to write

$$\int\int\int d^3\xi\ \ell_1\ell_2\ell_3\ \ (\nabla \cdot \mathbf{v}) = \sum_i \int d^3\xi\ \frac{\partial}{\partial \xi_i}\left(\frac{\ell_1\ell_2\ell_3}{\ell_i} v_i\right). \tag{73}$$

If we take the size of the rectangular region to zero, we see that the integrands are in fact equal. Therefore,

$$\nabla \cdot \mathbf{v} = \frac{1}{\ell_1\ell_2\ell_3} \sum_i \frac{\partial}{\partial \xi_i}\left(\frac{\ell_1\ell_2\ell_3}{\ell_i} v_i\right). \tag{74}$$

Finally, this result gives us

$$\nabla^2 \phi = \frac{1}{\ell_1\ell_2\ell_3} \sum_i \frac{\partial}{\partial \xi_i}\left(\frac{\ell_1\ell_2\ell_3}{\ell_i^2} \frac{\partial \phi}{\partial \xi_i}\right). \tag{75}$$

Example: spherical coordinates

As an example, first consider spherical coordinates. We have

$$\mathbf{r}(r,\theta,\varphi) = r\sin\theta\cos\varphi\hat{\mathbf{x}} + r\sin\theta\sin\varphi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}. \tag{76}$$

From this we compute $\ell_r = 1$, $\ell_\theta = r$ and $\ell_\varphi = r\sin\theta$. Therefore, we have

$$dV = drd\theta d\varphi\ r^2\sin\theta \tag{77}$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\phi}{\partial\varphi^2}. \tag{78}$$

Example: oblate spheroidal coordinates

The coordinate system (ξ,η,φ) with $0 \le \xi < \infty$, $-\pi/2 < \eta \le \pi/2$ and $0 \le \varphi < 2\pi$ defined by

$$\mathbf{r} = a\cosh\xi\cos\eta\cos\varphi\hat{\mathbf{x}} + a\cosh\xi\cos\eta\sin\varphi\hat{\mathbf{y}} + a\sinh\xi\sin\eta\hat{\mathbf{z}}. \tag{79}$$

Then

$$\begin{aligned} \ell_\xi &= \ell_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta} \\ \ell_\varphi &= a\cosh\xi\cos\eta. \end{aligned} \tag{80}$$

The Laplace's equation is then

$$\begin{aligned} 0 &= \frac{\partial}{\partial\xi}\left[\cosh\xi\cos\eta\frac{\partial\phi}{\partial\xi}\right] \\ &\quad+\frac{\partial}{\partial\eta}\left[\cosh\xi\cos\eta\frac{\partial\phi}{\partial\eta}\right] \\ &\quad+\frac{\partial}{\partial\varphi}\left[(\sinh^2\xi+\sin^2\eta)\frac{\partial\phi}{\partial\varphi}\right]. \end{aligned} \tag{81}$$

Consider the case of a conducting disk of radius 1 held at potential V. We'll look for a solution $\phi(\xi)$ since curves of constant ξ become a disk as $\xi \to 0$. The solution of Laplace's equation is then

$$\cosh\xi\frac{\partial\phi}{\partial\xi} = C \tag{82}$$

which implies

$$\phi = C_0 + 2C\tan^{-1}\left[\tanh\left(\frac{\xi}{2}\right)\right]. \tag{83}$$

When $\xi \to \infty$, $\phi \to C_0 + C\frac{\pi}{2}$. Since this should be zero, we have $C = -2C_0/\pi$. Finally, $\phi \to C_0 = V$ as $\xi \to 0$. Therefore,

$$\phi = V - \frac{4V}{\pi}\tan^{-1}\left[\tanh\left(\frac{\xi}{2}\right)\right]. \tag{84}$$

9.3 The curl

Finally, for the curl we use

$$\int d\mathbf{a}\cdot\nabla\times\mathbf{v} = \oint d\mathbf{l}\cdot\mathbf{v}. \tag{85}$$

Start with a square in $\vec{\xi}$ coordinates normal to ξ_3. Then

$$\int d\xi_1 d\xi_2\ \ell_1\ell_2[\nabla\times\mathbf{v}]_3 = \int d\xi_1 d\xi_2\frac{\partial}{\partial\xi_1}(\ell_2 v_2) - \int d\xi_1 d\xi_2\frac{\partial}{\partial\xi_2}(\ell_1 v_1). \tag{86}$$

After some computation, we obtain

$$\nabla\times\mathbf{v} = \frac{1}{\ell_1\ell_2\ell_3}\det\begin{pmatrix} \ell_1\hat{\mathbf{e}}_1 & \ell_2\hat{\mathbf{e}}_2 & \ell_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3} \\ \ell_1 v_1 & \ell_2 v_2 & \ell_3 v_3 \end{pmatrix} \tag{87}$$

9.4 Inversion through a sphere and the Kelvin transform

Consider the coordinate transformation

$$\mathbf{r}(\mathbf{R}) = a^2 \frac{\mathbf{R}}{|\mathbf{R}|^2}. \tag{88}$$

This mapping is called inversion because points inside a sphere of radius a get mapped to the region outside (the origin goes to infinity) while points outside get mapped to the inside of the sphere. The sphere of radius a reflected through the origin.

Consider a plane tangent to the sphere of radius a, described by $\mathbf{R} = a\hat{\mathbf{X}} + Y\hat{\mathbf{Y}} + Z\hat{\mathbf{Z}}$. This gets mapped to

$$\mathbf{r} = a^2 \frac{a\hat{\mathbf{X}} + Y\hat{\mathbf{Y}} + Z\hat{\mathbf{Z}}}{a^2 + Y^2 + Z^2}. \tag{89}$$

Note that

$$\left(\frac{a^3}{a^2 + Y^2 + Z^2} - \frac{a}{2}\right)^2 + \left(\frac{a^2 Y}{a^2 + Y^2 + Z^2}\right)^2 + \left(\frac{a^2 Z}{a^2 + Y^2 + Z^2}\right)^2 = a^2/4. \tag{90}$$

The plane, therefore, gets mapped to a sphere of radius $a/2$ at position $a/2\hat{\mathbf{X}}$. Notice also that this transformation is its own inverse. Therefore, the transformation also takes spheres in contact with the sphere of radius a to planes.

Then the basis vectors are

$$\frac{\partial \mathbf{r}}{\partial X} = a^2 \left[\frac{\hat{\mathbf{x}}}{|\mathbf{R}|^2} - \mathbf{R}\frac{2X}{|\mathbf{R}|^4}\right] \tag{91}$$

$$\frac{\partial \mathbf{r}}{\partial Y} = a^2 \left[\frac{\hat{\mathbf{y}}}{|\mathbf{R}|^2} - \mathbf{R}\frac{2Y}{|\mathbf{R}|^4}\right] \tag{92}$$

$$\frac{\partial \mathbf{r}}{\partial Z} = a^2 \left[\frac{\hat{\mathbf{z}}}{|\mathbf{R}|^2} - \mathbf{R}\frac{2Z}{|\mathbf{R}|^4}\right]. \tag{93}$$

These vectors are all mutually orthogonal. In addition,

$$\left(\frac{\partial \mathbf{r}}{\partial X}\right)^2 = \frac{a^4}{|\mathbf{R}|^4} \tag{94}$$

$$\left(\frac{\partial \mathbf{r}}{\partial Y}\right)^2 = \frac{a^4}{|\mathbf{R}|^4} \tag{95}$$

$$\left(\frac{\partial \mathbf{r}}{\partial Z}\right)^2 = \frac{a^4}{|\mathbf{R}|^4}. \tag{96}$$

So $\ell_1 = \ell_2 = \ell_3 = a^2/|\mathbf{R}|^2$. Therefore,

$$\nabla^2 \phi = \frac{|\mathbf{R}|^6}{a^6} \sum_i \frac{\partial}{\partial R_i}\left(\frac{a^2}{|\mathbf{R}|^2}\frac{\partial \phi}{\partial R_i}\right) \tag{97}$$

Finally, let $\phi = \tilde{\phi}|\mathbf{R}|$. Then

$$\begin{aligned}
\sum_i \frac{\partial}{\partial R_i}\left[|\mathbf{R}|^{-2}\frac{\partial \phi}{\partial R_i}\right] &= \sum_i \frac{\partial}{\partial R_i}\left[|\mathbf{R}|^{-1}\frac{\partial \tilde{\phi}}{\partial R_i} + R_i|\mathbf{R}|^{-3}\tilde{\phi}\right]. \\
&= \frac{1}{|\mathbf{R}|}\sum_i \frac{\partial^2 \tilde{\phi}}{\partial R_i^2}. \qquad (98)
\end{aligned}$$

Therefore, we have the result that – at least with the origin $|\mathbf{R}| = 0$ excluded (why?) –

$$\nabla^2 \phi = \frac{|\mathbf{R}|^5}{a^4}\sum_i \frac{\partial^2 \tilde{\phi}}{\partial R_i^2}. \qquad (99)$$

This defines the Kelvin transform:

$$\tilde{\phi}(\mathbf{R}) = \frac{1}{|\mathbf{R}|}\phi\left(\frac{a^2}{|\mathbf{R}|^2}\mathbf{R}\right). \qquad (100)$$

If $\tilde{\phi}$ solves Laplace's equation then so does $\phi(\mathbf{R})$. This is the three-dimensional analogue of the two-dimensional **conformal transformation**.

As an example, consider the problem of a charge q placed at a point $(x, y, z) = (0, 0, a)$ above a conducting plane held at zero potential. The solution is

$$\phi(\mathbf{r}) = \frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+a)^2}}. \qquad (101)$$

After a Kelvin transformation,

$$\begin{aligned}
\phi(\mathbf{r}) &= \frac{q}{|\vec{\xi}|}\left[\frac{1}{\sqrt{a^2 - 2a^3\xi_3/|\vec{\xi}|^2 + a^4/|\vec{\xi}|^2}} - \frac{1}{\sqrt{a^2 + 2a^3\xi_3/|\vec{\xi}|^2 + a^4/|\vec{\xi}|^2}}\right] \\
&= \frac{q}{a}\left[\frac{1}{\sqrt{|\vec{\xi}|^2 - 2a\xi_3 + a^2}} - \frac{1}{\sqrt{|\vec{\xi}|^2 + 2a\xi_3 + a^2}}\right]. \qquad (102)
\end{aligned}$$