Physics 605: Hilbert Spaces

Due: never

So what is wrong with an infinite-dimensional vector space? Well, nothing. But suppose we have an infinite number of basis elements, $\{|1\rangle, \cdots\}$. Then any vector is $|v\rangle = \sum_{i=1}^{\infty} c_i |i\rangle$ and, suddenly, we find ourselves dealing with infinite sequences. Therefore, it is important for us to worry about when sequences converge and when they don't. Since this is a first-year graduate course, we are not going to do a lot of worrying about this, but it is worth doing it a little in order to understand what is at stake.

1 Convergence of sequences

Convergence is one of those things physicists want to ignore because it's hard and complicated. Unfortunately, convergence is also important – it is the source of many errors in physics.

First, what do we mean by a sequence converging? Suppose we have a number x_n – it converges to x if the following this is true: for every $\epsilon > 0$, there is an N such that $|x_n - x| < \epsilon$ for any $n > N$. This gives us an intuitive notion of what it means for x_n to get closer and closer to x. Similarly, in a vector space we can use a norm to define convergence: for every $\epsilon > 0$, there is an N such that $\|v_n - v\| < \epsilon$ for any $n > N$.

Definition: A **Cauchy sequence** is one in which, for some ϵ, there is an N such that, if $n > N$ and an $m > N$, $\|v_n - v_m\| < \epsilon$. This just means elements in the sequence get closer and closer together.

Any convergent sequence is a Cauchy sequence; in order to get closer and closer to x, a sequence also has to get closer and closer to itself. So here is a question: is any Cauchy sequence also convergent? The answer, it turns out, is not necessarily, as can be seen by these two examples:

Counter-example: Here is an example based on an ancient Babylonian method to compute the square root of 2 using rational numbers. The vector space in question is the vector space of rational numbers (and the scalars are rational numbers).

Consider the sequence of rational numbers $x_{n+1} = (x_n + 2/x_n)/2$ with $x_0 =$

1. How do we figure out what it converges to? We look for fixed points: set $x_{n+1} = x_n = x_f$ and ask if there is a value such that the series does not grow or shrink. Algebra tells us that $x_f^2 = 2$ (or that $x_f = \pm\infty$). Now the question is to determine if this is a stable or unstable fixed point − does the sequence approach $\sqrt{2}$ or run away from it? The easiest way to do this is to consider $x_n = \sqrt{2} + \delta_n$ for some small δ. Then we have

$$\sqrt{2} + \delta_{n+1} \approx \sqrt{2} + \frac{\delta^2}{2\sqrt{2}} + \mathcal{O}(\delta^3). \tag{1}$$

If $\delta_n < 0$ then it's magnitude decreases. If $\delta_n > 0$, then its magnitude increases. We conclude that if $0 < x_0 < x_f$ then this sequence increases toward $\sqrt{2}$.

Now we know that the sequence gets closer and closer to $\sqrt{2}$. In fact, this sequence is Cauchy because the x_n must be getting closer to each other as they approach $\sqrt{2}$ and every element of x_n is rational. Yet it does not converge to a rational number!

Example: The real numbers are complete. In fact, one way to get the real numbers is to start with the rational numbers and *add* all the limits of every Cauchy sequence.

Definition: If all Cauchy sequences converge in a vector space, the space is called **complete** (or **Banach** if you want to be fancy about it). A complete space with an inner product is called a **Hilbert space**.

Interestingly, we can away take a Hilbert space and "complete" it. That is, given an infinite dimensional vector space, V, with inner product, we can form a Hilbert space, H, whose elements *are* the Cauchy sequences of V. The hard part is to prove that one can write an inner product for H using the inner product in V.

2 Dual spaces of Hilbert spaces

We learned that the dual space of an infinite dimensional vector space can be larger than the original vector space. What if our vector space is a Hilbert space?

Definition: The dual space, H^*, of an infinite-dimensional Hilbert space, H, is the vector space of *continuous* linear functionals on H. Linear functionals are continuous if and only if they are **bounded**, *i.e.* $|L\,|v\rangle\,| \leq M \|\,|v\rangle\,\|$ for some real number $M > 0$.

Definition: A **separable** Hilbert space is one in which every vector can be written as the limit of a Cauchy sequence. Interestingly, a Hilbert space is separable if and only if it has a countable number of basis vectors.

Riesz representation theorem: Let $L\,|v\rangle$ be a *bounded* linear functional (so it returns a scalar) and $|v\rangle$ be a vector in a separable Hilbert space. Then there is a $|w\rangle$ such that $L\,|v\rangle = \langle w|v\rangle$.

This tells us that almost everything we know is true about finite-dimensional vector spaces carries over to the case of separable Hilbert spaces. In particular, in finite-dimensional vector space, the dual space and vector space are isometric. Apparently, this is also true for Hilbert spaces.

Riesz-Fischer theorem: This theorem, proven in 1907, states that Hilbert space of functions, f, such that $\|f\|$ is finite, are also complete. They are not necessarily separable though.

3 Bases

Consider the Hilbert space of periodic functions $f(\theta)$. The set of functions

$$|n\rangle = f_n(\theta) = e^{in\theta} \tag{2}$$

are an orthonormal set of functions with $\langle f|g\rangle = \int_0^{2\pi} d\theta\; f^*(\theta)g(\theta)/(2\pi)$. These functions form a basis of the Hilbert space, meaning that

$$|v\rangle = \sum_n c_n\,|n\rangle = \sum_n |n\rangle\,\langle n|v\rangle \tag{3}$$

Notice how we can regroup this expression, $|v\rangle = (\sum_n |n\rangle\,\langle n|)\,|v\rangle$. We can think of

$$\mathbf{1} = \sum_n |n\rangle\,\langle n| \tag{4}$$

as a linear operator (equal to the identity). Let's rewrite this as functions:

$$v(\theta) = \sum_n e^{in\theta} \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{-in\theta'} v(\theta'). \tag{5}$$

This is really just the statement that Fourier series are invertible.

4 Dirac delta function

4.1 Basic definitions

Let's consider the space of periodic functions and the linear functional $L_0 |f\rangle = f(0)$. Is this linear functional in our new notion of a dual space? The answer is no – it isn't bounded because we can make $f(0)$ as large as we want while keeping $\|f\|$ finite. Hence, there can't be an M such that $|L_0 |f\rangle | \leq M\|f\|$. This is consistent with our notion that the Dirac delta function cannot really exist.

One approach to thinking about the delta function is to lift the restriction on bounded functionals. To do that, we want to also have a different notion of convergence.

Definition: Choose a suitable set of well-behaved, smooth (infinitely differentiable) functions, which we will call **test functions**. We say that $|v_n\rangle$ **converges weakly** to $|v\rangle$ if $\lim_{n\to\infty} \langle v_n|f\rangle = \langle v|f\rangle$ for every test function, f. Similarly, $|v_n\rangle$ **converges weakly** to L_0 if $\lim_{n\to\infty} \langle v_n|f\rangle = L_0 |f\rangle$.

Consider, for example, the sequence of functions

$$v_n(x) = \frac{1}{\sqrt{2\pi/n}} \exp\left(-n^2 \frac{x^2}{2}\right) \tag{6}$$

This sequence is not Cauchy, and shouldn't be part of a Hilbert space by all rights. Indeed, we have that $\lim_{n\to\infty} \langle v_n|f\rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} dx v_n^*(x) f(x) = f(0)$ which converges weakly but is not a bounded, linear functional.

Similar to how we handled completeness, one can create a new Hilbert space, sometimes called a **rigged Hilbert space** whose elements are the weakly-convergent series. This new, enlarged Hilbert space has a bunch of new elements in it, called **generalized functions** or **distributions**. One of these is the Dirac delta, $\delta(x)$. We might then say that $\lim_{n\to\infty} v_n(x) \sim \delta(x)$.

Sometimes people do the following calculation:

$$\int_{-Q}^{Q} dq\; e^{iqx} = \frac{2}{x} \sin(Qx). \tag{7}$$

They know (we will prove this soon) that $\int_{\infty}^{\infty} dq e^{iqx} = \delta(x)$ but they also see that, if they fix x, then $\sin(Qx)/x$ oscillates around zero without ever approaching it as $Q \to \infty$.

4.2 Other objects

What objects are in our rigged Hilbert space depends on what test functions we chose. Suppose our test functions are smooth. Then we can also make sense of distributions like $\partial^n \delta(x)/dx^n$ using integration-by-parts,

$$\int dx \; \frac{\partial^n \delta(x)}{\partial x^n} f(x) = (-1)^n \int dx \; \delta(x) \frac{\partial^n f}{\partial x^n} = (-1)^n \left. \frac{\partial^n f}{\partial x^n} \right|_{x=0}. \tag{8}$$

If our test functions could only be differentiated N times, then we could only define N derivatives of δ but not $N+1$.

4.3 Completeness relations

Consider

$$|v\rangle = \sum_n |n\rangle \langle n|v\rangle \tag{9}$$

For periodic functions, the basis set is $e^{im\theta}$ and Eq. (9) is equivalent to

$$\begin{aligned} v(\theta) &= \sum_n e^{in\theta} \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{-in\theta'} v(\theta') \\ &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \sum_n e^{in(\theta-\theta')} v(\theta'). \end{aligned} \tag{10}$$

Now we can ask the question, does $\sum_n e^{in(\theta-\theta')}$ converge? If it did, we would have $f(\theta-\theta') \equiv \sum_n e^{in(\theta-\theta')}$ so that

$$v(\theta) = \int_0^{2\pi} \frac{d\theta'}{2\pi} f(\theta-\theta') v(\theta'), \tag{11}$$

which implies that $f(\theta) = 2\pi\delta(\theta)$.

Of course, we switched the order of two limits here – the limit in the sum over n and the limit defining the integral. However, that is a bit of an illusion. We allow ourselves to switch the order symbolically but recall that the sequence itself is the element of the space. It's a bit of a mathematical technicality but it gets the job done; and that job is to allow you to pretend the manipulations you learned as undergraduates ok. Finally, we write

$$\sum_{n=-\infty}^{\infty} e^{in\theta} = 2\pi\delta(\theta) \tag{12}$$

converges weakly. Notice that this requires that θ lie between 0 and 2π to make sense.

4.4 Changes of variables

Suppose one has a function $f(x)$ with N isolated zeros, x_n, such that $f'(x_n) \neq 0$. Then

$$\delta[f(x)] = \sum_{n=1}^{N} \frac{\delta(x - x_n)}{|f'(x_n)|}. \tag{13}$$

As a consequence, consider the change of variables $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, and $z = r \cos\theta$ from spherical to cartesian coordinates. We have

$$\delta(x - x')\delta(y - y')\delta(z - z') = \frac{1}{(r')^2 \sin\theta'} \delta(r - r')\delta(\theta - \theta')\delta(\varphi - \varphi'). \tag{14}$$