Physics 605: Magnetostatics

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Maxwell's equations are

$$\begin{array}{cc} \nabla\cdot\mathbf{E} = 4\pi\rho & \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{array} \tag{1}$$

in *cgs* units. We also have the Lorenz force law,

$$\mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c}\times\mathbf{B}, \tag{2}$$

for a charge q moving with velocity v in an electric and magnetic field.

In the limit of electrostatics and magnetostatics, we assume that ρ, $\mathbf{E}$ and $\mathbf{B}$ do not change with time. In that case, we obtain

$$\begin{array}{cc} \nabla\cdot\mathbf{E} = 4\pi\rho & \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = 0 & \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} \end{array} \tag{3}$$

Note that, because $\nabla\cdot\nabla\times\mathbf{B} = 0$, we also require $\nabla\cdot\mathbf{J} = 0$. What does this mean about the electric currents? First, it means that $\partial\rho/\partial t = 0$ (by the continuity equation) – charges may be moving but there can be no accumulation or depletion of charge. One crucial consequence is that currents with $\nabla\cdot\mathbf{J} = 0$ must be closed loops.

The Helmholtz theorem applies here by telling us that $\mathbf{B} = \nabla\times\mathbf{A}$ where $\nabla\cdot\mathbf{A} = 0$. With this constraint, we have

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} \tag{4}$$

1 Biot-Savart Law

From Eq. (4), we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r'\;\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{5}$$

Taking the curl of both sides gives us our first important result: the **Biot-Savart** law. Indeed,

$$\begin{aligned} \mathbf{B} &= \nabla\times\mathbf{A} = \frac{1}{c}\int d^3r'\;\mathbf{J}(\mathbf{r}')\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{c}\int d^3r'\;\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \end{aligned} \tag{6}$$

In some sense, we are done - we can obtain the magnetic field from any distribution of currents. Of course, evaluating this integral is not always trivial.

The difficult part of using Eq. (6), aside from actually doing the integral, is thinking about what the current distribution is for a particular system.

2 Multipole Expansion

We'll start our analysis of magnetostatics with the multipole expansion associated with Eq. (4). Therefore, we use the Helmholtz solution

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{1}{c}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} &(7)\\ &= \frac{1}{c}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{2\ell+1}{4\pi}\frac{1}{r^{\ell+1}}Y_{\ell m}(\theta,\varphi)\int d^3r' \, \mathbf{J}(\mathbf{r}')(r')^{\ell}Y^*_{\ell m}(\theta',\varphi'). &(8)\end{aligned}$$

2.1 Monopole Term

The first term in the expansion is the monopole term,

$$\mathbf{A}_{(0)}(\mathbf{r}) = \frac{4\pi}{c|\mathbf{r}|}\int d^3r'\mathbf{J}(\mathbf{r}'). \tag{9}$$

This turns out to be zero, but it takes some work to prove it. To do so, consider

$$\sum_{j=1}^{3}\nabla'_j\cdot[\mathbf{r}'\mathbf{J}_j(\mathbf{r}')] = \mathbf{J}(\mathbf{r}')+\mathbf{r}'\nabla'\cdot\mathbf{J}(\mathbf{r}') = \mathbf{J}(\mathbf{r}'). \tag{10}$$

Therefore, we have an expression for the i^{th} component of $\mathbf{A}$ which is

$$\mathbf{A}_{i,(0)}(\mathbf{r}) = \frac{4\pi}{c|\mathbf{r}|}\int d^3r'\nabla'\cdot\left[\mathbf{r}'_i\mathbf{J}(\mathbf{r}')\right]. \tag{11}$$

By the divergence theorem, this evaluates to a surface integral at infinity, which vanishes under the assumption that $\mathbf{J}\to 0$ sufficiently fast. Therefore

$$\mathbf{A}_{i,(0)}(\mathbf{r}) = 0. \tag{12}$$

2.2 Dipole Term

The i^{th} component of the dipole term is

$$\mathbf{A}_{i,(1)} = \frac{4\pi}{c|\mathbf{r}|^3}\mathbf{r}\cdot\int d^3r' \, \mathbf{r}'J_i(\mathbf{r}'). \tag{13}$$

Lemma: If we define $m_{ij} = \int d^3r' \, r'_j J_i(\mathbf{r}')$ then $m_{ij} = -m_{ji}$.

To prove, consider

$$\sum_k \partial'_k \left(r'_i r'_j J_k \right) = r'_j J_i + r'_i J_j \tag{14}$$

since $\sum_k \partial'_k J_k = 0$. Integrating both sides and using the fact that the left-hand-side is a total divergence immediately gives us the result.

Lemma: The most general antisymmetric 3×3 matrix m_{ij} can be written as $m_{ij} = \sum_k \epsilon_{ijk} M_k$, where ϵ_{ijk} is the Levi-Civita antisymmetric tensor.

We will prove this directly by showing that $M_k = (1/2) \sum_{ij} \epsilon_{ijk} m_{ij}$. To do that, notice that

$$\sum_k \epsilon_{abk} M_k = \frac{1}{2} \sum_{kij} \epsilon_{abk} \epsilon_{ijk} m_{ij} = \frac{1}{2} \sum_{ij} \left(\delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi} \right) m_{ij},$$

using a standard identity $\sum_k \epsilon_{abk} \epsilon_{ijk} = \delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi}$. Then we have

$$\sum_k \epsilon_{abk} M_k = \frac{1}{2} \left(m_{ab} - m_{ba} \right) = m_{ab}.$$

The most general antisymmetric object can be written in terms of the completely antisymmetric tensor ϵ_{ijk} as $m_{ij} = \sum_k \epsilon_{ijk} M^k$ for some vector with components M^k. Then

$$\mathbf{A}_{i,(1)} = \frac{1}{c|\mathbf{r}|^3} \sum_j r_j \epsilon_{jik} M^k = \frac{1}{c|\mathbf{r}|^3} \mathbf{M} \times \mathbf{r}. \tag{15}$$

We call $\mathbf{M}$ the magnetic dipole moment of the current distribution.

in terms of the current, we then see that

$$M_k = \frac{1}{2} \sum_{ij} \epsilon_{ijk} m_{ij} = \frac{1}{2} \sum_{ij} \epsilon_{ijk} \int d^3 r' \; r'_i J_j(\mathbf{r}') \tag{16}$$

so

$$\mathbf{M} = \frac{1}{2} \int d^3 r' \; \mathbf{r}' \times \mathbf{J}(\mathbf{r}'). \tag{17}$$

3 Boundary conditions

Suppose there is a sheet of current. If this sheet lies along the $xy-$plane, for example, the current density could be

$$\mathbf{J} = \delta(z) \mathbf{K}, \tag{18}$$

where $\mathbf{K}$ is the **surface current density**. Notice that the units of $\mathbf{J}$ are charge/second/area; this implies the units of $\mathbf{K}$ are charge/second/length. Integrating the equation for magnetic field around a loop containing the surface immediately allows us to derive the following boundary conditions,.

$$\mathbf{B}_+ - \mathbf{B}_- = \frac{4\pi}{c}\mathbf{K} \times \mathbf{N}, \tag{19}$$

where $\mathbf{N}$ is the unit normal of the surface (pointing toward the + side). The normal component of $\mathbf{B}$, on the other hand, is continuous because $\nabla \cdot \mathbf{B} = 0$.

4 Magnetic Scalar Potential

Consider Maxwell's equations for $\mathbf{B}$ in a region without current. Then we have $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. In that case, we see immediately that $\mathbf{B} = -\nabla\phi_M$ for some scalar (and that $\nabla^2\phi_M = 0$). The scalar ϕ_M is called the **magnetic scalar potential**, and despite the really pointed similarities to the electric scalar potential, it is quite a different object.

For one thing, we know that $\mathbf{B} = -\nabla\phi$ but that, at a current carrying surface,

$$\mathbf{B}_+ - \mathbf{B}_- = \mathbf{K} \times \mathbf{n}. \tag{20}$$

This implies, in particular, that

$$-\nabla\phi_M\big|_+ + \nabla\phi_M\big|_- = \mathbf{K} \times \mathbf{n}. \tag{21}$$

This means that the normal component of $\nabla\phi_M$ is continuous across a surface of current. So is the component of $\nabla\phi_M$ directed along $\mathbf{K}$ on the surface. However, let $\mathbf{t}$ be a unit vector tangent to the surface but normal to $\mathbf{K}$. Therefore,

$$-\mathbf{t} \cdot \nabla\phi_M\big|_+ + \mathbf{t} \cdot \nabla\phi_M\big|_- = \mathbf{t} \cdot \mathbf{K} \times \mathbf{n} \tag{22}$$

If ϕ_M itself were continuous, then Eq. (20) could not be true – indeed, the two tangent components of $\nabla\phi$ would have to be continuous as well in that case.

4.1 Multiple-valuedness of ϕ_M

Let's consider a straight wire carrying a current. Ampere's law tells us that

$$\oint_L d\mathbf{l} \cdot \mathbf{B} = 4\pi I/c, \tag{23}$$

where I is the current enclosed by the loop L. However, since the $\mathbf{J} = 0$ everywhere outside the wire, we also have

$$\oint_L d\mathbf{l} \cdot \nabla \phi_M = -4\pi I/c. \tag{24}$$

How can this be? There is only one way: ϕ_M cannot be both singly-valued and continuous everywhere along the path. This makes it somewhat nontrivial to use as a mathematical tool but, nevertheless, still useful.

So what does ϕ_M look like for a straight wire? Taking advantage of the rotational symmetry about the wire and the translational symmetry along the wire, we assume $\mathbf{B} = B(r)\hat{\theta}$ and so

$$2\pi r \frac{\partial \phi_M}{\partial \theta} = -4\pi I/c. \tag{25}$$

Hence,

$$\phi_M = -2I(\theta - \theta_0)/c. \tag{26}$$

We explicitly see that a singly-valued ϕ_m jumps from $-2I(2\pi - \theta_0)/c$ to $2I\theta_0/c$ at $\theta = 0$. We can eliminate this discontinuity if we allow ϕ_M to be multiply-valued.

4.2 Example

To see this in action, let's work out an example. Consider a charged sphere of radius R, with surface charge density σ rotating with angular velocity ω. We can immediately write down $\mathbf{K}$ to be

$$\mathbf{K} = \sigma R \omega \sin\theta \hat{\phi}. \tag{27}$$

Then

$$-\nabla \phi_M\big|_+ + \nabla \phi_M\big|_- = \sigma \omega R \sin\theta \hat{\theta}. \tag{28}$$

Meanwhile, we can also write

$$\phi_M = \begin{cases} \sum_\ell a_\ell r^\ell P_\ell(\cos\theta), & r < R \\ \sum_\ell b_\ell r^{-\ell-1} P_\ell(\cos\theta), & r > R \end{cases}, \tag{29}$$

where we've taken advantage of azimuthal symmetry to simplify the expansion.

Therefore, we require

$$\begin{aligned} \sum_\ell a_\ell \ell R^{\ell-1} P_\ell(\cos\theta) + \sum_\ell b_\ell (\ell+1) R^{-\ell-2} P_\ell(\cos\theta) &= 0 \qquad (30) \\ \sum_\ell \left[a_\ell R^\ell \sin\theta - b_\ell R^{-\ell-1} \sin\theta\right] P_\ell'(\cos\theta) &= \sigma\omega R \sin\theta. \end{aligned}$$

The first equation tells us that $b_\ell = \ell R^{2\ell+1} a_\ell/(\ell+1)$. The second equation can be rewritten as

$$\frac{\partial}{\partial\theta}\left[\sum_\ell \left[a_\ell R^\ell - b_\ell R^{-\ell-1}\right] P_\ell(\cos\theta) + \sigma\omega R\cos\theta\right] = 0. \tag{31}$$

Therefore, we may suppose that

$$\sum_\ell \left[a_\ell R^\ell - b_\ell R^{-\ell-1}\right] P_\ell(\cos\theta) + \sigma\omega R\cos\theta = C_0. \tag{32}$$

Then we have

$$\begin{aligned} C_0\delta_{\ell,0}l &= a_\ell R^\ell - b_\ell R^{-\ell-1} + \sigma\omega R\delta_{\ell,1} \\ &= a_\ell R^\ell \frac{1}{\ell+1} + \sigma\omega R\delta_{\ell,1}. \end{aligned} \tag{33}$$

Therefore,

$$a_\ell = -2\sigma\omega\delta_{\ell,1} + C_0 \tag{34}$$

$$b_\ell = -\sigma\omega\delta_{\ell,1}. \tag{35}$$

Therefore,

$$\phi_M = \begin{cases} C_0 - 2\sigma\omega r\cos\theta, & r < R \\ -\sigma\omega\frac{1}{r^2}\cos\theta, & r > R. \end{cases} \tag{36}$$

5 Magnetic Materials

We start with the notion of the magnetization, $\mathbf{M}$, the magnetic dipole moment per unit volume in a material. We start by expanding $\mathbf{M}$ in powers of $\mathbf{B}$, assuming that it depends only locally on the magnetic field. This gives us

$$\mathbf{M} = \mathbf{M}_0 + \chi\mathbf{B}, \tag{37}$$

where χ is the magnetic susceptibility tensor. Here we are explicitly including the possibility that a material has a permanent magnetic dipole. In general, most materials do not and so we will assume $\mathbf{M}_0 = 0$ in most of the following development.

The vector potential can be written using

$$\begin{aligned} \mathbf{A} &= \int d^3r'\, \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{c|\mathbf{r}-\mathbf{r}'|} = \int d^3r'\, \mathbf{M}(\mathbf{r}') \times \nabla\frac{1}{c|\mathbf{r}-\mathbf{r}'|} \\ &= -\int d^3r' \nabla' \times \left[\frac{M(\mathbf{r}')}{c|\mathbf{r}-\mathbf{r}'|}\right] + \int d^3r' \frac{\nabla' \times M(\mathbf{r}')}{c|\mathbf{r}-\mathbf{r}'|}. \end{aligned} \tag{38}$$

The first term is a total derivative, but it's not obvious how to express it that way as it is written. Let's write it the value of the i^{th} components explicitly using indices. This is

$$\begin{aligned}
-\int d^3r'\nabla' \times \left[\frac{M(\mathbf{r}')}{c|\mathbf{r}-\mathbf{r}'|}\right]_i &= -\sum_{jk}\int d^3r'\ \partial'_j\left[\epsilon_{ijk}M_k(\mathbf{r}')\frac{1}{c|\mathbf{r}-\mathbf{r}'|}\right] \\
&= \int d^3r' \sum_j \partial'_j\left[\sum_k \epsilon_{jik}M_k(\mathbf{r}')\frac{1}{c|\mathbf{r}-\mathbf{r}'|}\right] \quad (39)\\
&= \sum_{jk}\oint d\mathbf{a}_j \epsilon_{jik}M_k(\mathbf{r}')\frac{1}{c|\mathbf{r}-\mathbf{r}'|} \\
&= -\sum_{jk}\oint d\mathbf{a}_j \epsilon_{ijk}M_k(\mathbf{r}')\frac{1}{c|\mathbf{r}-\mathbf{r}'|} \quad (40)
\end{aligned}$$

By the standard assumptions, this integral vanishes. Therefore,

$$-\nabla^2\mathbf{A} = \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{4\pi}{c}\nabla\times\mathbf{M} \quad (41)$$

This can then be rewritten by introducing $\mathbf{H} = \mathbf{B} - (4\pi/c)\mathbf{M}$ so that

$$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J}. \quad (42)$$

Notice that $\nabla\cdot\mathbf{H} = -4\pi\nabla\cdot\mathbf{M}/c \neq 0$.

For historical reasons, we will actually write $\mathbf{M} = \chi_m\mathbf{H}$ so that, if we define $\mu = (\mathbf{1} + 4\pi\chi/c)$, then $\mathbf{B} = \mu\mathbf{H}$. The tensor μ is the magnetic permeability of the material. We will almost always assume, as is typical, that μ is proportional to the identity.

5.1 Torque on a magnetic moment

As one possible, entirely classical model of how the magnetization may arise, let's consider a material made up of many small loops of current. Think, for example, of the electrons whizzing around the nucleus of an atom if you must. Let's write the current of one such loop as

$$\mathbf{I} = I\left(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right), \quad (43)$$

where θ is the cylindrical coordinate azimuthal angle. The magnetic moment of this current loop is

$$\mathbf{m} = \frac{1}{c}\int d\theta R\ \mathbf{r}\times\mathbf{I} = \frac{\pi R^2 I}{c}\hat{\mathbf{z}}. \quad (44)$$

We can compute the total force on this current loop and the total torque using a differential version of the Lorentz force law. In particular, the force per unit length on the loop is

$$\mathbf{f} = \frac{1}{c}\mathbf{I} \times \mathbf{B}. \tag{45}$$

Therefore,

$$\begin{aligned}
\mathbf{F} &= \frac{I}{c}\int_0^{2\pi} d\theta R\ (-\sin\theta B_y \hat{\mathbf{z}} + \sin\theta B_z \hat{\mathbf{y}} \qquad (46)\\
&\quad -\cos\theta B_x \hat{\mathbf{z}} + \cos\theta B_z \hat{\mathbf{x}}) = 0.\\
\mathbf{T} &= \frac{IR^2}{c}\int_0^{2\pi} d\theta \left(-B_y \hat{\mathbf{x}} \cos^2\theta + B_x \hat{\mathbf{y}} \sin^2\theta\right) + \cdots\\
&= |\mathbf{m}| \left(-B_y \hat{\mathbf{x}} + B_x \hat{\mathbf{y}}\right)\\
&= \mathbf{m} \times \mathbf{B}, \qquad (47)
\end{aligned}$$

where the $\cdots$ are terms that will eventually integrate to zero.

The equilibrium position occurs when $\mathbf{m} \propto \mathbf{B}$. Indeed, the only *stable* equilibrium occurs when $\mathbf{m}$ and $\mathbf{B}$ point in the same direction. Consequently, we can write a formula for the equilibrium magnetic dipole moment, $\mathbf{m}(\mathbf{B}) = m\mathbf{B}$. A material made up of these dipole moments will then have

$$\mathbf{M} = M\mathbf{B}. \tag{48}$$

Writing $\mathbf{H} = (1 + 4\pi M/c)\,\mathbf{B}$ we can then rewrite the formula for the magnetization as

$$\mathbf{M}(\mathbf{H}) = \frac{M}{1 + 4\pi M/c}\mathbf{H}. \tag{49}$$

Don't assume, however, that $\chi = M/(1 + 4\pi M/c) > 0$ is always true. More generally, one requires quantum mechanics and statistical mechanics to do this calculation more rigorously. It turns out that we can have either $\chi > 0$ or $\chi < 0$. The former case is called a **diamagnet**, the latter a **paramagnet**. Moreover, the interaction between neighboring magnetic moments can facilitate a "permanent" magnetic dipole (think of iron, for example).

6 The Gauss Integral

Let's think about Ampere's law for a loop of current, L_1,

$$4\pi IN/c = \oint_{L_2} d\mathbf{l}_2 \cdot \mathbf{B}. \tag{50}$$

The loop L_2 must wind around the current loop L_1 obviously, and does so N times. Now substitute in the Biot-Savart law,

$$4\pi IN/c = \frac{I}{c} \oint_{L_2} \oint_{L_1} d\mathbf{l}_2 \cdot \left(d\mathbf{l}_1 \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right). \tag{51}$$

Canceling out the I/c and rearranging a little gives us a result first noted by Gauss,

$$N = \frac{1}{4\pi} \oint_{L_1} \oint_{L_2} d\mathbf{l}_1 \times d\mathbf{l}_2 \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{52}$$

Apparently, given two loops that wind around each other N times, this integral is always equal to precisely N. This is known as an index theorem – an integral of a continuous function that, somehow, depends only on an integer that is determined only by the topology of some geometrical object.