Physics 605: Special Functions as solutions to differential equations

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Here we are going to work out the eigenfunctions of the Laplacian in 2 and 3 dimensions in a few different domains. What kind of boundary conditions should we impose? Let's consider an inner product,

$$\langle f|g\rangle = \int_{\mathbf{D}} d^D\mathbf{r} f^*(\mathbf{r})g(\mathbf{r}), \tag{1}$$

over some domain $\mathcal{D}$ in D dimensions. Under what conditions is the Laplacian self-adjoint?

Let's just compute it directly:

$$\begin{aligned} \langle f|\nabla^2 g\rangle &= \int_{\mathbf{D}} d^D\mathbf{r} f^*(\mathbf{r})\nabla^2 g(\mathbf{r}) \\ &= \int_{\partial\mathbf{D}} d^{D-1}\mathbf{a}\cdot\nabla g(\mathbf{r})\ f^*(\mathbf{r}) - \int_{\mathcal{D}} d^D\mathbf{r}\nabla f^*\cdot\nabla g \\ &= \int_{\partial\mathbf{D}} d^{D-1}\mathbf{a}\cdot\nabla g(\mathbf{r})\ f^*(\mathbf{r}) - \int_{\partial\mathbf{D}} d^{D-1}\mathbf{a}\cdot\nabla f^*(\mathbf{r})\ g(\mathbf{r}) + \langle\nabla^2 f|g\rangle\,. \end{aligned} \tag{2}$$

Eq. (2) implies that the eigenvalues of ∇^2 are negative in any dimension. This calculation shows that either

$$\hat{\mathbf{N}}\cdot\nabla\phi = 0 \quad \text{or} \quad \phi = 0 \tag{3}$$

on the boundary, where $\hat{\mathbf{N}}$ is the *outward* pointing unit normal to the region $\mathcal{D}$.

1 2D Rectangular Domain

Let's first find the eigenfunctions in a rectangular domain with periodic boundary conditions $\phi(x, L) = \phi(x, 0)$ and $\phi(L, y) = \phi(0, y)$. Then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -k^2\phi. \tag{4}$$

The orthonormal eigenfunctions are

$$|n, m\rangle = \frac{1}{L}e^{i2\pi nx/L}e^{i2\pi my/L} = \frac{1}{L}e^{i\mathbf{k}_{nm}\cdot\mathbf{r}}, \tag{5}$$

where $\mathbf{k}_{nm} \equiv 2\pi nx/L\hat{\mathbf{x}} + 2\pi my/L\hat{\mathbf{y}}$.

Notice that the functions $|n, m\rangle$ are actually simultaneous eigenfunctions of the Laplacian and ∂_x^2 (and ∂_y^2). This is reasonable because, in fact,

$$\begin{aligned} \left[\nabla^2, \partial_x^2\right] &\equiv \nabla^2 \partial_x^2 - \partial_x^2 \nabla^2 = 0 \\ \left[\nabla^2, \partial_y^2\right] &= 0 \\ \left[\partial_x^2, \partial_y^2\right] &= 0. \end{aligned}$$

Thus, these operators are all simultaneously diagonalizable.

1.1 Completeness relations

The eigenfunctions of ∇^2 form a basis for the Hilbert space of functions that vanish on the rectangular domain with periodic boundary conditions. This allows us to compute a number of identities associated with the completeness of this basis. For instance,

$$\begin{aligned} |v\rangle &= \sum_{nm} |n, m\rangle \langle n, m|v\rangle \\ &= \sum_{n,m} \frac{1}{L^2} e^{i\mathbf{k}_{nm}\cdot\mathbf{r}} \int d^2\mathbf{r}'\ e^{-i\mathbf{k}_{nm}\cdot\mathbf{r}'} v(\mathbf{r}'). \end{aligned} \tag{6}$$

Now let's allow functions which converge only weakly (see the notes on Hilbert spaces for more information about this). Then we can expand our Hilbert space to include objects like the Dirac delta. In such a space, we can perform the following manipulation,

$$|v\rangle = \frac{1}{L^2} \int d^2\mathbf{r}' \left[\sum_{nm} e^{i\mathbf{k}_{nm}\cdot(\mathbf{r}-\mathbf{r}')}\right] v(\mathbf{r}'). \tag{7}$$

Thus, we conclude that

$$\sum_{nm} e^{i\mathbf{k}_{nm}\cdot(\mathbf{r}-\mathbf{r}')} = L^2 \delta^2(\mathbf{r}-\mathbf{r}') = L^2 \delta(x-x')\delta(y-y') \tag{8}$$

This complements the orthonormality relation,

$$\delta_{nn'}\delta_{mm'} = \frac{1}{L^2} \int d^2\mathbf{r}\ e^{i(\mathbf{k}_{nm}-\mathbf{k}_{n'm'})\cdot\mathbf{r}} \tag{9}$$

2 2D Disk

Consider the two-dimensional circular domain, $r < R$, and the Hilbert space of smooth functions that vanish on $r = R$. The eigenvalue problem for the

Laplacian in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=-k^2\phi. \tag{10}$$

We seek to find simultaneous eigenfunctions of ∇^2 and ∂_θ^2. To do so, we notice that $[\nabla^2, \partial_\theta^2]$ indeed commute.

We first expand in the θ coordinate by writing

$$\phi(r,\theta)=\sum_{m=-\infty}^{\infty} c_m(r)e^{im\theta}. \tag{11}$$

Therefore we find

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c_m}{\partial r}\right)-\frac{m^2}{r^2}c_m+k^2c_m=0. \tag{12}$$

This is a Sturm-Liousville problem with $W(r)=r$. This is called **Bessel**'s equation. What do its solutions look like? We know that there will be two solutions since it is a second-order equation. Let's call them J_m and Y_m. Indeed,

$$\phi=\sum_{m=-\infty}^{\infty}\left[\tilde{c}_m J_m(kr)+\tilde{d}_m Y_m(kr)\right]e^{im\theta}. \tag{13}$$

These solutions are called **Bessel** functions. Ultimately, our job is to discover the properties of these functions. Once we know the properties, we can use those properties to manipulate them the same way we manipulate sin and cos.

2.1 Bessel's equation

Traditionally, Bessel's equation is generalized somewhat,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)-\frac{\nu^2}{r^2}f+k^2f=0, \tag{14}$$

where ν is allowed to be any real number.

2.2 Large r

When r is large, the equation is approximately

$$\frac{\partial^2}{\partial r^2}f+k^2f\approx 0. \tag{15}$$

The solutions are therefore $f\approx A\sin(kr)+B\cos(kr)$ when $r\to\infty$. These functions are oscillatory for large r and, therefore, have an infinite number of zeros.

2.3 Series expansion

One way to attempt a solution is as a series expansion in r. Technically, this is called the **Frobenius** method. Write

$$c_m(r) = r^\alpha \sum_{n=0}^{\infty} a_n r^n \tag{16}$$

and substitute it into the equation.

Then

$$\begin{aligned} 0 &= a_0\left(\alpha^2 - m^2\right) r^{\alpha-2} a_1[(\alpha+1)^2 - m^2] r^{\alpha-1} \\ &\quad + \sum_{n=2}^{\infty}\left[a_n(\alpha+n)^2 - a_n m^2 + k^2 a_{n-2}\right] r^{\alpha+n-2}. \end{aligned} \tag{17}$$

The solution arises by setting the coefficients of the powers of r to zero.

Now we have some choices to make. Clearly either a_0 or a_1 must be nonzero. Suppose $a_0 \neq 0$. Then $\alpha = \pm m$ and $a_1 = 0$.

2.3.1 Bessel functions of the first kind

If we choose $\alpha = m$, which is consistent with a smooth function through the disk, we obtain the recursion relation,

$$a_n = \frac{-k^2}{(\alpha+n)^2 - m^2} a_{n-2} = \frac{k^2 a_{n-2}}{n(n+2m)}, \tag{18}$$

setting the other coefficients. Let's call the resulting function $J_m(kr)$. It is clear that these solutions to Bessel's equation do not have any singularities in them.

A more useful formula for J_m comes from a **generating function**. Indeed, one can show that

$$G_J(x,z) = \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x) z^m. \tag{19}$$

This is useful because we can use $G_J(x,z)$ to prove statements about all the Bessel functions J_m, in one fell swoop. We can show from direct substitution that

$$\frac{1}{r}\partial_r\left(r\partial_r G_J(kr,z)\right) + k^2 G_J(kr,z) = \frac{1}{r^2} z\partial_z\left(z\partial_z G_J(kr,z)\right). \tag{20}$$

Now we expand $G_J(x,z)$ in powers of z using the general expansion of Eq. (19). Notice that

$$z\partial_z\left(z\partial_z G_J\right) = \sum_{m=-\infty}^{\infty} m^2 J_m(x) z^m. \tag{21}$$

Therefore,

$$\left\{\frac{1}{r}\partial_r\left[r\partial_r J_m(kr)\right] + \left(k^2 - \frac{m^2}{r^2}\right)J_m(kr)\right\}z^m = 0. \tag{22}$$

Setting like powers of z^m equal implies that the coefficients of G_J corresponding to different powers of m do, indeed, solve Bessel's equation.

An example of something we can now prove is that $J_m(0) = \delta_{m0}$. Simply consider the expansion of $G_J(0,z)$ in powers of z, which yields $G_J(0,z) = 1$. Since this is also equal to $G_J(0,z) = \sum_{m=-\infty}^{\infty} J_m(0)z^m$, we conclude that only $J_0(0)$ can be nonzero (and is, in fact, 1). A less trivial result can be derived from

$$\frac{\partial}{\partial x}G_J(x,z) = \frac{1}{2}\left(z - \frac{1}{z}\right)G_J(x,z). \tag{23}$$

Putting both terms on one side and expanding both sides in powers of z, we obtain

$$\sum_{m=-\infty}^{\infty} z^m\left\{J_m'(x) - \frac{1}{2}\left[J_{m-1}(x) - J_{m+1}\right]\right\} = 0. \tag{24}$$

Hence $2J_m'(x) = J_{m-1}(x) + J_{m+1}(x)$.

2.3.2 Bessel functions of the second kind

When $\alpha = -m$, we have functions with singularities as $r \to 0$. We can call these function J_{-m}. Sadly, it ends up being proportional to J_m when m is an integer. We can see this directly from the generating function $G_J(x,z)$. Notice that $G_J(-x, 1/z) = G_J(x,z)$. Expanding both sides of this equation in powers of z yields

$$\sum_{m=-\infty}^{\infty}\left[J_{-m}(-x) - J_m(x)\right]z^m = 0 \tag{25}$$

which must vanish term by term. Consequently, $J_{-m}(x) = J_m(-x)$. It turns out that this is a consequence of m being an integer. If we look at solutions of the generalized Bessel equation with non-integer m, we instead obtain that $J_\nu(x)$ and $J_{-\nu}(x)$ are, indeed, linearly independent. So what do we do when ν is an integer? Instead, we define

$$Y_\nu(kr) = \frac{J_\nu(kr)\cos(\nu\pi) - J_{-\nu}(kr)}{\sin(\nu\pi)} \tag{26}$$

and $Y_m(kr)$ becomes Y_ν as $\nu \to m$. Notice that Y_ν inherits the divergences of $J_{-\nu}$. In addition, Y_0 turns out to diverge logarithmically (whereas $J_{-\nu}$ does not).

2.4 Applying the boundary conditions

Now we know that Y_m diverges as $r \to 0$ so that $\tilde{d}_m = 0$. Let κ_{mn} be the n^{th} zero of J_m. Then the remaining boundary condition at $r = R$ tells us that

$$|n, m\rangle = J_m(\kappa_{mn} r/R) e^{im\theta} \tag{27}$$

is the most general eigenfunction of the Laplacian in the disk $r < R$ that vanishes on the boundary. Any function satisfying those boundary conditions can now be written as a linear combination,

$$f(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} c_{mn} J_m(\kappa_{mn} r/R) e^{im\theta} \tag{28}$$

The orthogonality relation for $J_m(kr)$ is

$$\int_0^R dr\ r\ J_m(\kappa_{mn} r/R) J_m(\kappa_{n'm'} r/R) = \frac{1}{2} \delta_{nn'} \int_0^R dr r J_m^2(\kappa_{nm} r/R). \tag{29}$$

It turns out that you can evaluate the integral on the right hand side in terms of Bessel functions. Indeed, one can show that

$$\begin{aligned} \int_0^R dr\ r J_m^2(\kappa_{nm} r/R) &= -\frac{R^2}{2} J_{m-1}(\kappa_{nm}) J_{m+1}(\kappa_{nm}) \\ &= -\frac{R^2}{2} J_{m-1}^2(\kappa_{nm}) \end{aligned} \tag{30}$$

2.5 Completeness relations

Now we obtain the completeness relationship,

$$2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{J_m(\kappa_{mn} r/R) J_m(\kappa_{mn} r'/R)}{R^2 J_{m-1}^2(\kappa_{nm})} = \frac{1}{r} \delta(r - r'). \tag{31}$$

2.6 Integral form

To find an integral form for solutions, we start with the solution in cartesian coordinates. These are

$$e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{32}$$

We wish to expand these in the orthonormal $e^{im\theta}/\sqrt{2\pi}$ – the bases vectors for the azimuthal part of the solution. To do this, note that

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_m e^{im\theta} \int \frac{d\theta}{2\pi} e^{i|\mathbf{k}| r \cos(\theta - \theta_0) - im\theta}, \tag{33}$$

which is just a consequence that $e^{im\theta}$ form an orthonormal basis using the correct inner product. This equation just expresses $e^{i\mathbf{k}\cdot\mathbf{r}}$ in terms of this basis. Of course, since these are periodic in θ, the value of θ_0 does not matter. Consequently,

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_m e^{im\theta} \int \frac{d\theta}{2\pi} e^{ikr\cos\theta - im\theta}, \tag{34}$$

where $k = |\mathbf{k}|$.

Since the left-hand side of Eq. (34) is an eigenfunction of ∇^2 then the right-hand side also is. Therefore, the terms on the right-hand side must be linear combinations of J_m and Y_m. If we take $r \to 0$, we see that these functions are all non-singular. Therefore, it *must* be true that

$$J_m(kr) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ikr\cos\theta - im\theta}. \tag{35}$$

Strictly speaking, we only know these are proportional but can match the expansion of $J_m(kr)$ in powers of r with our previous expansion to prove this equation is actually an equality.

3 Laplacian on a sphere

> This material briefly discusses the context of Chapter 12 of Arfken and Weber.

In spherical coordinates,

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right]. \tag{36}$$

The portion independent of r is

$$-\mathbf{L}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}, \tag{37}$$

is the Laplacian on the surface of a sphere, parametrized by (θ, φ). The strange minus sign in Eq. (37) is there so that the eigenvalues of $\mathbf{L}^2$ end up being positive. Here, we are going to find the eigenfunctions and eigenvalues of $\mathbf{L}^2$.

3.1 The long way around

The eigenvector equation takes the form

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = -\lambda\phi. \tag{38}$$

Next we write $\phi = \sum_m c_m e^{im\varphi}$ so that

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c_m}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}c_m = -\lambda c_m. \tag{39}$$

If $x = \cos\theta$ then $\partial/\partial\theta = (\partial x/\partial\theta)\partial/\partial x = -\sin\theta\partial/\partial x$. Therefore,

$$\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial c_m}{\partial x}\right] - \frac{m^2}{1-x^2}c_m = -\lambda c_m. \tag{40}$$

Notice that something funny happens at the poles of the sphere, when $\theta = 0$ or π. However, the $c_m(\theta)$ must not be singular at the poles. It turns out that this requires $\lambda = \ell(\ell+1)$ for integers $\ell \geq 0$. This is, by no means, obvious – it takes quite a bit of work to show. The solutions to $\mathbf{L}^2 Y_{lm} = \ell(\ell+1)Y_{lm}$ are called **spherical harmonics**. We will see why this is true in the next section.

Let's see if we can solve for for $m = 0$. Then we have

$$\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial c_0}{\partial x}\right] = -\lambda c_0. \tag{41}$$

The solutions to this equation are

$$P_\ell(x) = \frac{1}{2^\ell\ \ell!}\frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right] \tag{42}$$

with eigenvalues $\lambda = \ell(\ell+1)$. The $P_\ell(x)$ are the Legendre polynomials. We can compute the first few as

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= \frac{1}{2}\frac{d}{dx}\left(x^2-1\right) = x \\ P_2(x) &= \frac{1}{4\times 2}\frac{d^2}{dx^2}\left(x^4-2x^2+1\right) = \frac{3x^2-1}{2}, \end{aligned}$$

and so on. These are defined so that $P_\ell(1) = 1$.

How else can we see this? We note that this is a Sturm-Liousville problem with $W = \sin\theta$. The inner product is

$$\langle f|g\rangle = \int_{-\pi}^{\pi} d\theta \sin\theta f(\theta)g(\theta) = \int_{-1}^{1} dx\ f(x)g(x). \tag{43}$$

If we try to solve our equation with polynomials in x, we notice that those polynomials must be orthogonal to each other. We've already seen a bunch of polynomials orthogonal with this inner product – the Legendre polynomials. And indeed, Eq. (42) give a formula for the Legendre polynomials.

3.2 Angular momentum

The operator $\mathbf{L}^2$ also describes the angular momentum in quantum mechanics. Let's use this to our advantage by decomposing it into its three components, L_x, L_y and L_z. Indeed, $\mathbf{L} = -i\mathbf{r} \times \nabla$ gives us precisely a first-order operator whose square is $\mathbf{L}^2$. Let's compute it explicitly in cartesian coordinates to make sure. We will use components and the Levi-Cevita tensor ϵ_{ijk} (which is 1 for ϵ_{123} and completely antisymmetric otherwise). Then we have

$$\begin{aligned} \mathbf{L}^2 &= -\sum_{ijkab} \epsilon_{ijk} r_j \partial_k \left(\epsilon_{iab} r_a \partial_b\right) \\ &\quad -\sum_{ijkab} \epsilon_{ijk}\epsilon_{iab} r_j \partial_k \left(r_a \partial_b\right) \end{aligned} \tag{44}$$

Now we note that $\sum_i \epsilon_{ijk}\epsilon_{iab} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}$ and so

$$\mathbf{L}^2 = -\sum_{jkab} \left(\delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}\right) r_j \partial_k \left(r_a \partial_b\right) \tag{45}$$

We can also show that $L_z = -i\partial/\partial\phi$.

The operators satisfy the following commutation relations,

$$[L_x, L_y] = iL_z \tag{46}$$

$$\left[\mathbf{L}^2, L_I\right] = 0,\ I = x, y,\ \text{or } z \tag{47}$$

Since they commute, we know that we can simultaneously diagonalize $\mathbf{L}^2$ and L_z. That is what we will do. Let $|Y\rangle$ be a simultaneous eigenvector of both $\mathbf{L}^2$ and L_z.

Let's also define $L_\pm = L_x \pm iL_y$. Then

$$[L_z, L_\pm] = \pm L_\pm \tag{48}$$

$$[L_+, L_-] = 2L_z. \tag{49}$$

Suppose that $L_z |Y\rangle = m |Y\rangle$ (since it is an eigenvector). Then, since $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$, we know that the eigenvalues of $\mathbf{L}^2$ are strictly larger than m^2.

Furthermore, $L_z L_+ |m\rangle = L_+ L_z |m\rangle + [L_z, L_+] |m\rangle = (m+1)|m\rangle$. So

$$L_+ |m\rangle \propto |m+1\rangle \tag{50}$$

$$L_- |m\rangle \propto |m-1\rangle \tag{51}$$

Thus, we call $L_\pm$ the ladder operators.

Now what happens if we take $(L_+)^k |m\rangle \propto |m+k\rangle$. But m cannot grow without bound so there must be some k such that $L_+^k |m\rangle = 0$. Let k be the smallest integer such that this is true. Then, $L_-L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + i[L_x, L_y] = \mathbf{L}^2 - L_z^2 - L_z$. In that case, $\mathbf{L}^2 = L_-L_+ + L_z^2 + L_z$. Then $L_-(L_+)^k |m\rangle = 0 = [\mathbf{L}^2 - (m+k)^2 - (m+k)]L_+^{k-1} |m\rangle$. Define $\ell = m+k$. Then $\mathbf{L}^2 = \ell(\ell+1)$. We also discover, from this, that $|m| \leq \ell$.

Therefore, $\mathbf{L}^2 |\ell, m\rangle = \ell(\ell+1) |\ell, m\rangle$ and the eigenfunctions are indexed by $|\ell, m\rangle$ where $|m| \leq \ell$. The orthonormalized eigenfunctions will be $Y_{\ell m}(\theta, \phi)$. How do we actually find them? Applying $L_\pm$ to the solutions we found with $m = 0$.

3.3 Orthogonality

The most important thing to know about the spherical harmonics is

$$\int d\Omega \; Y_{\ell m}^*(\theta,\varphi) Y_{\ell' m'}(\theta,\varphi) \quad = \quad \frac{4\pi}{2\ell+1} \delta_{mm'} \delta_{\ell\ell'}, \tag{52}$$

where $d\Omega = d\varphi d\theta \sin\theta$. Of course, they are orthogonal, as they must be as they have either a different eigenvalue with $\mathbf{L}^2$ or L_z. The factor of $4\pi/(2\ell+1)$ on the right-hand side can be taken as the definition of the normalization of the spherical harmonics.

3.4 Legendre polynomials

The Legendre polynomials arise as spherical harmonics with $m = 0$. In particular, $P_\ell(\cos\theta) \propto Y_{\ell 0}(\theta, \varphi)$. Traditionally, the normalization is set at $\theta = 0$, where $\cos\theta = 1$, by $P_\ell(1) = 1$. Therefore, we see that $P_0 = 1$, $P_1 = x$, $P_2 = (3x^2-1)/2$, and so on.

4 Constructive Method for Obtaining the Spherical Harmonics

4.1 Circular Harmonics

There is another method to obtain the spherical harmonics that is constructive. We start with the definition of a homogeneous polynomial.

Definition: A **homogeneous polynomial**, $p_\ell(x_1, x_2, \cdots, x_n)$ of degree ℓ satisfies $p_\ell(rx_1, \cdots, rx_n) = r^\ell p_\ell(x_1, \cdots, x_n)$. An example would be, $p_2(x, y) = x^2 + xy$.

Homogenous polynomials of degree ℓ form a vector space. The **harmonic**, homogeneous polynomials of degree n, defined by $\nabla^2 p_\ell(x_1, \cdots) = 0$, also make up a vector space.

In 2D, for example, we can look at the space of harmonic, homogenous polynomials $p_\ell(x, y)$. We have

$\ell = 0$: The vectors $\{1\}$ form a basis.

$\ell = 1$: The vectors $\{x, y\}$ form a basis.

$\ell = 2$: The vectors $\{x^2, xy, y^2\}$ form a basis for the homogeneous polynomials. However, $\{x^2/2 - y^2/2, xy\}$ form a basis for the harmonic, homogeneous polynomials.

$\ell = 3$: The harmonic, homogeneous polynomials are spanned by the basis $\{x^3/3 - xy^2, y^3/3 - yx^2\}$.

Now, consider the fact that $\nabla^2 p_\ell(x, y) = 0$ in polar coordinates. We write $x = r\cos\theta$ and $y = r\sin\theta$ so that $p_\ell(x, y) = r^\ell p_\ell(\cos\theta, \sin\theta)$, which implies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial r^\ell}{\partial r}\right)p_\ell(\cos\theta, \sin\theta) + \frac{r^\ell}{r^2}\frac{\partial^2 p_\ell(\cos\theta, \sin\theta)}{\partial\theta^2} = 0. \tag{53}$$

Hence,

$$\frac{\partial^2 p_\ell(\cos\theta, \sin\theta)}{\partial\theta^2} = -\ell^2 p_\ell(\cos\theta, \sin\theta). \tag{54}$$

Hence, the homogeneous, harmonic polynomials can also be used to form the eigenvectors of *part* of the Laplacian. Indeed, we notice that

$\ell = 0$: $\{1\}$

$\ell = 1$: $\{\cos\theta, \sin\theta\}$

$\ell = 2$: $\{\cos(2\theta), \sin(2\theta)\}$

and so on are just the sines and cosines that make up the basis for the space of periodic functions!

4.2 Spherical harmonics

Now we think about the harmonic, homogeneous polynomials $p_\ell(x,y,z)$ with $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$. We have

$$0 = \nabla^2 p_\ell = \frac{1}{r^2}\left(r^2\frac{\partial r^\ell}{\partial r}\right)p_\ell + r^{\ell-2}\mathbf{L}^2 p_\ell. \quad (55)$$

Hence, we see that

$$\mathbf{L}^2 p_\ell = -\ell(\ell+1)p_\ell. \quad (56)$$

Let's look at these polynomials for $r = 1$. Then

$$\begin{aligned}
\ell &= 0: \{1\} \\
\ell &= 1: \{x,y,z\} = \{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\} \\
\ell &= 2: \{x^2-y^2, x^2-z^2, xy, xz, yz\} = \\
&\quad \{\sin^2\theta\cos(2\varphi), \sin^2(\theta)\cos^2(\varphi)-\cos^2(\theta), \sin^2(\theta)\sin(2\varphi), \sin(2\theta)\cos\varphi, \sin(2\theta)\sin\varphi\}.
\end{aligned}$$

Since these functions all have the same eigenvalue, there is no guarantee that they are orthogonal with any inner product. And indeed, they are not. However, since $\mathbf{L}^2$ and $L_z = -i\partial/\partial\varphi$ commute, we can use L_z to diagonalize the functions within each subspace of ℓ. Consequently, we arrive at the spherical harmonics (or functions proportional to them anyway):

$$\begin{aligned}
\ell &= 0: \{1\} \\
\ell &= 1: \{\sin\theta e^{i\varphi}, \sin\theta e^{-i\varphi}, \cos\theta\} \\
\ell &= 2: \{\sin^2\theta e^{2i\varphi}, 3\cos\theta - 1, \sin^2(\theta)e^{-2i\varphi}, \sin(2\theta)e^{i\varphi}, \sin(2\theta)e^{-i\varphi}\}.
\end{aligned}$$

Finally, we choose

$$\int d\Omega\; Y^*_{\ell m}(\theta,\varphi)Y_{\ell' m'}(\theta,\varphi) = \frac{4\pi}{2\ell+1}\delta_{mm'}\delta_{\ell\ell'}, \quad (57)$$

where $d\Omega = d\varphi d\theta\sin\theta$.

5 Harmonic functions in spherical coordinates

Any solution to Laplace's equation can be expanded in spherical harmonics. In other words,

$$\phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} R_{\ell m}(r)Y_{\ell m}(\theta,\varphi). \quad (58)$$

When the solution is axisymmetric, then we can alternately expand in $P_\ell(cos\theta)$. Therefore,

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{\ell m}}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}R_{\ell m} = 0 \tag{59}$$

and we find that $R_{\ell m} = A_{\ell m}r^\ell + B_{\ell m}r^{-(\ell+1)}$. Similarly, for axisymmetric solutions,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)P_\ell(\cos\theta). \tag{60}$$

These expansions are called the **multipole expansion**, however the method is valid for any equation whose solutions can be expanded in $Y_{\ell m}$.

These formulas play an important role in solving electrostatics problems, since spherical geometries are so common.