Physics 605: Approximation Methods

Due: never

1 Method of steepest descent

(Arfken & Weber pg. 489)

Consider the integral for the Gamma function

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}. \tag{1}$$

This function has the important property that $\Gamma(z+1) = z\Gamma(z)$ so that, when z is an integer, n, then $\Gamma(n) = (n-1)!$. Notice, however, that $\Gamma(z)$ is defined on the complex plane. For this to make sense, we must interpret $t^{z-1} = e^{(z-1)\ln t}$. Incidentally, Gauss invented something called $\Pi(z) = \int_0^\infty dt t^z e^{-t}$ which is an all-around better function that $\Gamma(z)$. Like Betamax$^{(\mathrm{TM})}$ (Google it!), it failed to catch on even though it was a superior product.

To learn the method of steepest descent, we will consider $\Gamma(n+1)$ for large, real n. Then we have

$$\Gamma(n+1) = \int_0^\infty dt\ t^n e^{-t} = \int_0^\infty dt\ e^{-t+n\ln t}. \tag{2}$$

Let's rescale t in the following way: let $t = n\xi$. Then

$$\Gamma(n) = n^{n-1} \int_0^\infty d\xi\ e^{-n(\xi - \ln \xi)}. \tag{3}$$

Our interest is to do this modified integral when we know that n is large. In general, the method of steepest-descent will allow us to handle integrals that can be written as

$$I = \int_C dz f(z) e^{-ng(z)}, \tag{4}$$

where $f(z)$ does not depend exponentially on n, which often arise in quantum mechanics and statistical mechanics.

The method of steepest-descent takes advantage of the contour-independence of these integrals. The trick is to deform the contour in such a way as to allow as to make it easy to approximate the integral when n is large.

1.1 Saddle points

A saddle point is defined as a point in which $\partial g(z)|_{z=z_0} = 0$. If $g(z)$ is holomorphic in the vicinity of such a point then we have, necessarily,

$$g(z) \approx g(z_0) + \frac{1}{2} g''(z_0)(z - z_0)^2 + \cdots \,, \tag{5}$$

where we assume that $g''(z_0)$ is small. In fact, the **Morse lemma** ensures that every function $g(z)$ is infinitesimally close to one in which $g''(z_0)$ is nonzero so this is not too large of a constraint.

Since this is a complex function, asking whether the saddle-point at z_0 is a maximum or minimum doesn't quite make sense. We can ask, however, what $g(z)$ looks like near its saddle-point in terms of $z = x + iy$. Let $\delta z = \delta x + i\delta y = z - z_0$. Then we have

$$\mathrm{Re}\ g(z) \approx \mathrm{Re}\ g(z_0) + \frac{1}{2}\mathrm{Re}\ g''(z_0)(\delta x^2 - \delta y^2) - \mathrm{Im}\ g''(z_0)\delta x \delta y \tag{6}$$

$$\mathrm{Im}\ g(z) \approx \mathrm{Im}\ g(z_0) + \frac{1}{2}\mathrm{Im}\ g''(z_0)(\delta x^2 - \delta y^2) + \mathrm{Re}\ g''(z_0)\delta x \delta y. \tag{7}$$

Near a saddle-point, part of the integrand looks like

$$\begin{aligned} e^{-ng(z)} \approx\ & e^{-ng(z_0)} e^{-n\mathrm{Re}g''(z_0)\ (\delta x^2 - \delta y^2)/2 + n\mathrm{Im}g''(z_0)\ \delta x \delta y} \\ & \times e^{-ni\mathrm{Im}g''(z_0)\ (\delta x^2 - \delta y^2) - ni\mathrm{Re}g''(z_0)\ \delta x \delta y}. \end{aligned} \tag{8}$$

The good news is that, not only can we deform a contour to pass through a saddle-point, we can also choose which direction it travels in as it passes. Indeed, suppose we choose our contour to pass through z_0 and to do so in such a way that the imaginary component of $g''(z_0)(z - z_0)^2 - g(z_0)$ vanishes. Then

$$\frac{1}{2}\mathrm{Im}\ g''(z_0)(\delta x^2 - \delta y^2) + \mathrm{Re}\ g''(z_0)\delta x \delta y = 0 \tag{9}$$

Let $\delta x = t\cos\theta$ and $\delta y = t\sin\theta$. Then this equation becomes

$$\mathrm{Im}\ g''(z_0)\cos(2\theta) + \mathrm{Re}\ g''(z_0)\sin(2\theta) = 0, \tag{10}$$

or

$$-\mathrm{Im}\ g''(z_0)/\mathrm{Re}\ g''(z_0) = \tan(2\theta). \tag{11}$$

Substituting everything back into the integrand, in the vicinity of the saddle-point we have

$$e^{-ng(z)} \approx e^{-ng(z_0)} \exp\left\{-\frac{n}{2}t^2 \left[\mathrm{Re}g''(z_0)\ \cos(2\theta) - \mathrm{Im}g''(z_0)\ \sin(2\theta)\right]\right\} \tag{12}$$

With our choice of θ, the argument of the exponential decreases rapidly as t increases. The vicinity of the saddle-point itself seems to dominate the integrand. Indeed,

$$e^{-ng(z)} \quad \approx \quad e^{-ng(z_0)} \exp\left\{-\frac{n}{2}t^2|g''(z_0)|\right\}. \tag{13}$$

1.2 Approximating integrals near saddle-points

As n becomes larger and larger, the saddle-points of $g(z)$ end up dominating the entire integral. We write

$$\begin{aligned} I &= \int_C dz f(z) e^{-ng(z)} \\ &\approx e^{i\theta} \int dt\ f(z_0) e^{-ng(z_0)} \exp\left[-\frac{n}{2}t^2|g''(z_0)|\right]. \end{aligned} \tag{14}$$

For large n, we may as well take the limits of the integral from $-\infty$ to ∞ since the integral is dominated by the saddle point anyway. Then we have

$$I \approx e^{i\theta} f(z_0) e^{-ng(z_0)} \sqrt{\frac{2\pi}{n|g''(z_0)|}}. \tag{15}$$

If $g(z)$ has more than one saddle-point, then the integral is a sum over the contributions of each of the individual saddle-points.

2 Stirling's Approximation

Now let's go back to our problem of approximating $\Gamma(n+1)$ for large n. We have

$$n! = \Gamma(n+1) = n^{n+1} \int_0^\infty d\xi e^{-n(\xi - \ln \xi)}. \tag{16}$$

Let $g(z) = z - \ln z$. Then $g'(z) = 1 - 1/z$ so $g'(z_0) = 0$ is solved by $z_0 = 1$. Then we have $g(1) = 1$ and $g''(1) = 1$. From our analysis, it seems clear that $\theta = 0$ (the contour deforms to be parallel to the real axis) and so,

$$n! = \Gamma(n+1) \approx n^n e^{-n} \sqrt{2\pi n}. \tag{17}$$

This is called **Stirling's approximation** for the factorial.