Physics 605: Complex Variables

Due: never

The complex numbers can be thought of - formally - as a space of numbers made of ordered pairs (x, y) such that

1. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
2. $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$.

These two operations both commute and are associative. Indeed, the complex numbers are an example of a mathematical structure called a **field**.

The space of complex numbers is probably one of the most remarkable things in math. First, it contains the real numbers inside it: the subspace of numbers $(x_1, 0)$ behave precisely like the real numbers. There is a number, $i = (0, 1)$ such that $i^2 = (-1, 0) = -1$. This gives us an alternate notation to use for any complex number, $z = x + iy$, which recognizes the fact that $(1, 0)$ and $(0, 1)$ form a basis for the vector space of ordered pairs.

There is also a particularly nice linear map from the **complex plane** to itself called **complex conjugation** – $z^* = x - iy$. This is geometrically a mirror reflection through the real axis, so it leaves the real axis unharmed. Interestingly, $z^* z$ is a real number. We define $|z| = \sqrt{z^* z}$ as the magnitude of the complex number z.

The complex numbers arose in mathematics because any n^{th} degree polynomial $p(z) = 0$ has precisely n complex roots. There is another representation of the complex numbers that indicate that, even without the polynomials, we'd end up thinking about them anyway. Consider the space spanned by the two matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

This space is isomorphic to the complex numbers, where matrix multiplication plays the role of complex multiplication.

1 Functions of a complex variable

Any complex function of two real variables, $f(x, y)$, can be re-expressed as a function of $z = x + iy$. By convention, we write $f(z, z^*)$ to retain the sense that these are functions of two arguments rather than one. Moreover, we can construct any function of x and y with combinations of z and z^*. Then we can write

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \tag{2}$$

and

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] \tag{3}$$

In complex analysis, we specialize to the case of functions of the form $f(z)$. That is, functions with the property $\partial f/\partial z^* = 0$. If we think of $f = u + iv$, then we can decompress this equation to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{4}$$

These equations are called the Cauchy-Riemann equations. Any differentiable function satisfying these equations is called holomorphic.

The Cauchy-Riemann conditions are highly restrictive. Notice that,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{5}$$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y}. \tag{6}$$

Therefore,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0. \tag{7}$$

So the function u is harmonic. A similar calculation shows that v is also a harmonic function. The two functions u and v are known as **conjugate-harmonic**.

We can get a more geometrical handle on what these equations mean if we think about the function $f(z)$ as a *vector* function. Define $\mathbf{f} = (u, -v)$. Then

$$\nabla \cdot \mathbf{f} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, \tag{8}$$

But the right-hand side precisely vanishes by the Cauchy-Riemann equations. Hence $\mathbf{f}$ is a divergence free vector field. Going further, compute the 2D curl of $\mathbf{f}$,

$$\nabla \times \mathbf{f} = \epsilon_{ij}\partial_i \mathbf{f}_j = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{9}$$

So holomorphic functions are vector fields that are entirely divergence-free and curl-free in two dimensions.

1.1 A few words about the two-dimensional curl

Notice that the curl in 2D is a scalar. This means that it can also be rewritten as a divergence of a different vector field. To wit, rewrite the curl as $\epsilon_{ij}\partial_i \mathbf{f}_j = \partial_i (\epsilon_{ij}\mathbf{f}_j)$. If we define $\mathbf{F}_i = \epsilon_{ij}\mathbf{f}_j = (-v, -u)$, we see that $\nabla \times \mathbf{f} = \nabla \cdot \mathbf{F}$. What is $\mathbf{F}$? It is just $\mathbf{f}$ rotated by ninety degrees at each point.

While these relations are entirely unrelated to $\mathbf{f}$ corresponding to a holomorphic function, I find it satisfying that $\mathbf{F}$ is the vector field representation of $if(z) = -v + iu$. Since $\mathbf{f}$ is divergence- and curl-free for holomorphic functions, we conclude that $\mathbf{F}$ is also divergence- and curl-free.

2 Analytic functions

An analytic function is one that can be expressed as a power series that converges in some open subset. A necessary condition to have an analytic function is that it must have an infinite number of derivatives (that is, all analytic functions are smooth). For real functions,

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{10}$$

is analytic if the series converges around the point x_0.

As we saw on a homework about bump functions, in real space the converse need not be true: there are plenty of smooth but not analytic functions. Similarly, we define an analytic, complex function as one that can be written as

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \tag{11}$$

where the series converges in some domain of the complex plane.

One amazing theorem – maybe the most important of complex analysis – is this: **Theorem:** (Goursat) Functions that are holomorphic in a domain D are also analytic in that domain.

One simple consequence of this is that, if you can take one derivative of a function $f(z)$ and the those derivatives satisfy the Cauchy-Riemann equations, then you can take an infinite number of them. Strangely, the proof requires us to understand something how to integrate complex functions.

3 Properties of holomorphic functions

We know that complex functions can be viewed as a special case of a divergence- and curl-free vector field. To integrate vector field's on the plane, we can compute a line integral along a path $\mathbf{R}(s)$,

$$\int_{\mathbf{R}} d\mathbf{l} \cdot \mathbf{f} = \int_a^b ds\ \frac{\partial \mathbf{R}}{\partial s} \cdot \mathbf{f}[\mathbf{R}(s)]. \tag{12}$$

To integrate a function on the complex plane, we need to specify a contour, $z(s)$, on the complex plane as well. We then define

$$\int_C dz f(z) \equiv \int_a^b ds \frac{\partial z(s)}{\partial s} f[z(s)]. \tag{13}$$

If we take this seriously, let's write $f = u + iv$ and $z = x + iy$. Then

$$\begin{aligned} \int_C dz f(z) &= \int_a^b ds \left\{ \frac{\partial x(s)}{\partial s} u[z(s)] - \frac{\partial y(s)}{\partial s} v[z(s)] \right\} \\ &\quad + i \int_a^b ds \left\{ \frac{\partial x(s)}{\partial s} v[z(s)] + \frac{\partial y(s)}{\partial s} u[z(s)] \right\}. \end{aligned} \tag{14}$$

This seems reminiscent of the formula for the line integral but, also, different. Still, it is instructive to try to rewrite this as 2D path integrals. To do that, let's take $\partial_s z(s) \to \partial_s \mathbf{R} = (\partial_s x, \partial_s y)$. Then we see that

$$\begin{aligned} \int_C dz f(z) &= \int_a^b ds \partial_s \mathbf{R} \cdot \mathbf{f}[\mathbf{R}(s)] + i \int_a^b ds \partial_s \mathbf{R} \times \mathbf{f}[\mathbf{R}(s)] \\ &= \int_a^b ds \partial_s \mathbf{R} \cdot \mathbf{f}[\mathbf{R}(s)] + i \int_a^b ds \partial_s \mathbf{R} \cdot \mathbf{F}[\mathbf{R}(s)] \end{aligned} \tag{15}$$

So the real and imaginary parts are just path integrals of the two vector fields $\mathbf{f}$ and $\mathbf{F}$.

This leads us to the critically important theorem,

Theorem: (Cauchy) Let $f(z)$ be holomorphic in a domain, $\mathcal{D}$, and C be any closed contour entirely within $\mathcal{D}$ surrounding a point w on the complex plane. Then

$$\oint_C dz\ f(z) = 0.$$

The proof is a consequence of the fact that the line integral around a closed path of a curl-free vector field is always zero. At this point, it seems as if holomorphic functions are boring. The above result can be parlayed into a more useful result.

Corollary: Let $f(z)$ be holomorphic in a domain, $\mathcal{D}$, and C be any closed contour entirely within $\mathcal{D}$ surrounding a point w on the complex plane. Then

$$\oint_C dz\ \frac{f(z)}{z-\omega} = 2\pi i f(w).$$

Instead of a proof, let's assume the result of the integral is path independent (as long as the contour contains point w) and consider a particular contour: a circle of radius r with the point w at its center. Then $z(\theta) = w + re^{i\theta}$ and we have

$$\begin{aligned} \oint_C dz\ \frac{f(z)}{z-\omega} &= \int_0^{2\pi} d\theta re^{i\theta} i \frac{f(w+re^{i\theta})}{re^{i\theta}} \\ &= i\int_0^{2\pi} d\theta f(w+re^{i\theta}). \end{aligned} \tag{16}$$

Taking $r \to 0$, we see that this gives $2\pi i f(w)$.

But why stop there?

Corollary: Let $f(z)$ be holomorphic in a domain, $\mathcal{D}$, and C be any closed contour entirely within $\mathcal{D}$ surrounding a point w on the complex plane. Then

$$\frac{n!}{2\pi i}\oint_C dz\ \frac{f(z)}{(z-\omega)^{n+1}} = f^{(n)}(w).$$

Because of this formula, any holomorphic function has an infinite number of derivatives; and we can compute them by doing integrals! Indeed,

Theorem: (Goursat) A function holomorphic in a domain $\mathcal{D}$ is also analytic in that domain.

To prove this result, we would simply construct the expansion we want – the Taylor expansion – and write

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(w)(z-w)^n. \tag{17}$$

Replacing the derivatives with their respective integral formulas, we obtain a series expansion for any holomorphic function. The crux of the proof of Goursat's theorem then involves showing that the series expansion thus obtained converges to the original holomorphic function.

3.1 Entire Functions

Entire function: An entire function is one for which the Taylor expansion converges on the entire complex plane.

If a holomorphic function $f(z)$ is not entire, then it has a finite **radius of convergence**. That is, its Taylor expansion converges inside an open disk (the disk without the boundary) of radius R.

Let's write a Taylor expansion for an arbitrary entire function by expanding around the origin. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{18}$$

For this to converge everywhere on the *entire* complex plane (get it? get it?) we need

$$\lim_{n\to\infty} |a_n|^{1/n} = 0, \tag{19}$$

which just means the coefficients do not grow too rapidly.

Entire functions are useful, in particular, because they actually happen. One important example is the exponential function, $e^z = \sum_{n=0}^{\infty} z^n/n!$.

Theorem: (Louisville's theorem) Any bounded, entire function must be constant.

To prove this, write $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \frac{1}{2\pi i}\oint_C dz \frac{f(z)}{z^{n+1}} \tag{20}$$

and consider a circular contour of radius r containing the origin. Then we have

$$|a_n| = \left|\frac{r^{-n}}{2\pi}\int d\theta f(re^{i\theta})e^{-in\theta}\right| \leq \frac{r^{-n}}{2\pi}\int d\theta |f(re^{i\theta})|. \tag{21}$$

Since the function is bounded, we also know that $|f(re^{i\theta})| \leq M$ for all z. Therefore, $a_n \leq Mr^{-n}$. Letting $r \to \infty$ shows that $a_n \leq 0$ for all $n \neq 0$. Since the contour integral can't depend on the radius of the contour, it must be that $a_n = 0$ for all $n \neq 0$. Hence, the function is constant.

Theorem: (Maximum modulus principle) If $f(z)$ is a holomorphic function then $|f(z)|$ has no maximum.

Theorem: (Weierstrass Factorization Theorem) Any entire function can be written as a product of its zeros. In other words, we can always write any entire function as

$$f(z) = cz^{m_0}e^{g(z)} \prod_{n, z_n \neq 0} E_{p_n}(z/z_n), \tag{22}$$

where

$$E_p(z) = \begin{cases} (1-z), & p = 0 \\ (1-z)\exp[\sum_{n=1}^{p} z^n/n], & p > 0 \end{cases} \tag{23}$$

p_n is a sequence of integers, $g(z)$ is another entire function, $f(z_n) = 0$ and $f(z)$ has no other zeros. When there is a zero at $z = 0$, we set $m > 0$. Remarkably, this is true for even infinite products. Note that $e^{g(z)}$ has no zeros – that's why I wrote it that way.

Not complete: We can use this to develop some nutty infinite products of entire functions. Let's take the function $sin(\pi z)$ as an example. The function $\sin(\pi z)$ has roots on the real axis at $z = n$ for any integer n. Therefore, we immediately see that

$$\sin(\pi z) = e^{g(z)} z^m \prod_{n \neq 0} E_{p_n}(z/n). \tag{24}$$

This leaves us with the task of finding the sequence p_n and the entire function $f(z)$. If we look at the Taylor expansion of $\sin(\pi z)$ near $z = 0$, we see that $m = 1$ and $g(z) = \ln \pi + \mathcal{O}(z)$. Similarly, let's consider the Taylor expansion of $\sin(\pi z)$ near the pole at n. This is

$$\sin(\pi z) \approx \pi(z-n) - \frac{1}{6}\pi^3(z-n)^3 + \cdots \tag{25}$$

Now we can compare this to the expansion of our infinite product. Note that $E_{p_m}(z/m) \approx e^{n+n^2/2+\cdots}$ when $m \neq n$. Therefore, we have

$$e^{g(z)} z \prod_{n \neq 0} E_{p_n}(z/n) \approx e^{g(n)}(n-z)e^{\sum_{m=1}^{p_n} n^m/m} + \cdots \tag{26}$$

4 Meromorphic Functions

A meromorphic function is holomorphic within a domain $\mathcal{D}$ except for at isolated points. It turns out that any meromorphic function is the ratio of two analytic functions. Suppose the meromorphic function $f(z)$ has a pole at a point w. Then we can write any meromorphic function using a **Laurent series**,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-w)^n \tag{27}$$

where

$$a_n = \frac{1}{2\pi i} \oint dz \frac{f(z)}{(z-w)^{n+1}}, \tag{28}$$

where the integral is taken over a path containing w but no other non-holomorphic points. If $f(z)$ is meromorphic, then there is an $N < 0$ such that $a_N \neq 0$. If this happens, we say that $f(z)$ has a pole at w, and if $a_N \neq 0$ for $N < 0$ but $a_n = 0$ for $n < N$ then we say w is an N^{th} order pole.

5 More complex contour integrals

Let's suppose we have a meromorphic function

$$f(z) = \frac{g(z)}{\prod_{m=1}^{M} (z - z_m)}, \tag{29}$$

where $g(z)$ is analytic. Suppose we choose a contour C containing all M simple

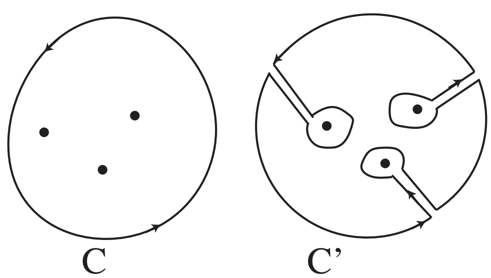

Figure 1: Contours

poles. Consider the slightly different contour, C', which contains no poles (see figure 1). The contour C' decomposes into a region equivalent to C, several long straight lines and circular contours around each pole. In the limit that $\epsilon \to 0$, we therefore have

$$0 = \int_{C'} dz f(z) = \int_C dz f(z) - \sum_{m=1}^{M} \int_{C_m} dz f(z), \tag{30}$$

Consequently,

$$\int_{C'} dz f(z) = \int_C dz f(z) = \sum_{m=1}^{M} \int_{C_m} dz f(z). \tag{31}$$

The right-hand side can be evaluated by

$$\begin{aligned} \int_{C'} dz f(z) &= \int_C dz f(z) = \sum_{m=1}^{M} \int_{C_m} dz \frac{g(z)}{\prod_{n\neq m}(z-z_n)} \frac{1}{z-z_n} \\ &= 2\pi i \sum_{m=1}^{M} \frac{g(z_m)}{\prod_{n\neq m}(z_m - z_n)}. \end{aligned} \tag{32}$$

because

$$\frac{g(z)}{\prod_{n\neq m}(z-z_n)} \tag{33}$$

is analytic in the domain containing the contour C_m.

We can formalize this result a little more with the following definition:

Definition: A **residue** of $f(z)$ at z_n for an m^{th} order pole is given by

$$\mathrm{Res}_{z=z_n} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_n} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_n)^m f(z)\right]. \tag{34}$$

Therefore, in this language we have

$$\oint_C \frac{dz}{2\pi i} f(z) = \sum_m \mathrm{Res}_{z=z_m} f(z), \tag{35}$$

where the sum is over the poles contained.

5.1 Example 1

Consider the integral

$$I = \int_{-\pi}^{\pi} d\theta \; \frac{1}{1+3cos^2\theta}. \tag{36}$$

Wd can rewrite this integral as a contour integral for a specific contour, the unit circle defined by $z(\theta) = e^{i\theta}$. In particular, note that $\partial_\theta z = d\theta i e^{i\theta} = d\theta i z$ so that

$$\begin{aligned} I &= \int_C \frac{dz}{iz} \frac{1}{1+3/4(z+1/z)^2} \\ &= \frac{4}{3} \int_C \frac{dz}{i} \frac{z}{z^4 + 10/3z^2 + 1}. \end{aligned} \tag{37}$$

The integrand is a meromorphic function with poles defined by

$$z^2 = -\frac{5}{3} \pm \sqrt{25/9 - 9/9} = -\frac{5}{3} \pm \frac{4}{3} = -\frac{1}{3} \text{ or } -3. \tag{38}$$

Therefore, $z = \pm i/\sqrt{3}$ and $z = \pm i\sqrt{3}$ are the four poles. Two of these poles are contained within the unit circle. Therefore,

$$I = \frac{8\pi}{3}\left[\frac{-i/3}{(3+1/3)(-2i/3)} + \frac{i/3}{(3+1/3)(2i/3)}\right] = \frac{8\pi}{3}\frac{3}{10} = \frac{4\pi}{5}. \tag{39}$$

5.2 Example 2

Another integral is

$$I = \lim_{R\to\infty}\int_{-R}^{R}\frac{dk}{2\pi}\frac{e^{ixk}}{k^2+m^2}. \tag{40}$$

This two can be written as a contour integral using the contour $z(k) = k$. Then we have

$$I = \int_C \frac{dz}{2\pi}\frac{e^{ixz}}{(z-im)(z+im)}. \tag{41}$$

To do this integral, we consider a modifying our contour by adding a second contour, which we call C_R, defined by $z(\theta) = Re^{i\theta}$, where θ ranges from 0 to π. Thus, this arc lives entirely in the upper half of the complex plane. In particular, $\mathrm{Im}z(\theta) > 0$. Therefore,

$$\begin{aligned}\int_{C+C_R}\frac{dz}{2\pi}\frac{e^{ixz}}{(z-im)(z+im)} &= \int_C \frac{dz}{2\pi}\frac{e^{ixz}}{(z-im)(z+im)} \\ &\quad + \int_{C_R}\frac{dz}{2\pi}\frac{e^{ixz}}{(z-im)(z+im)}.\end{aligned} \tag{42}$$

Note that

$$\int_{C_R}\frac{dz}{2\pi}\frac{e^{ixz}}{(z-im)(z+im)} = \int_0^\pi \frac{d\theta}{2\pi} iRe^{i\theta}\frac{e^{ixRe^{i\theta}}}{R^2e^{2i\theta}+m^2}. \tag{43}$$

As long as $x > 0$, the integrand vanishes as $R \to \infty$. Therefore,

$$\lim_{R\to\infty}\int_{C+C_R}\frac{dz}{2\pi}\frac{e^{ixz}}{(z-im)(z+im)} = I. \tag{44}$$

Since the contour contains the pole $z = im$, we have

$$I = ie^{-xm}\frac{1}{2im} = \frac{1}{2m}e^{-xm},\ x > 0. \tag{45}$$

If $x < 0$, we instead have to choose our contour $z(\theta) = Re^{i\theta}$ where θ ranges from π to 2π. Then $\mathrm{Im} z(\theta) < 0$. Then we obtain

$$I = \frac{1}{2m} e^{xm}. \tag{46}$$

Together, we see that

$$I = \frac{1}{2m} e^{-m|x|}. \tag{47}$$

5.3 Antiderivatives

In real analysis, we have

$$\int_0^x dy\ f'(y) = f(x) - f(0). \tag{48}$$

That is, derivatives and integrals are, sort of, inverses of each other. In complex analysis we have contour integrals and derivatives of holomorphic functions. The question is, is there a similar statement in complex analysis.

Let $f(z)$ be a holomorphic function in a domain $\mathcal{D}$ and let $F(z)$ be a holomorphic function such that $\partial F(z) = f(z)$. Then let's compute

$$\int_C dz f(z) \tag{49}$$

for a contour between points z_1 and z_2. In particular, let the contour range from $z(\xi_1) = z_1$ to $z(\xi_2) = z_2$. Then

$$\int_C dz f(z) = \int_{\xi_1}^{\xi_2} d\xi \partial_\xi z(\xi) f[z(\xi)] = \int_{\xi_1}^{\xi_2} d\xi \partial_\xi \{F[z(\xi)]\} = F(z_2) - F(z_1). \tag{50}$$

6 Principle Value

Let's suppose $f(z)$ is meromorphic and $|f(z)| \to 0$ at infinite (in the upper half plane). Then consider

$$I = \int_{-\infty}^{\infty} dx\ \frac{f(z)}{x - x_0}, \tag{51}$$

where x and x_0 is real. If x_0 wasn't on the real axis, we could close the contour in the upper half plane and do this as a contour integral. But it appears that we cannot – at last not obviously. The only reasonable solution is to push the pole slightly off the contour (well, any other solution that makes sense is equivalent to this anyway). Let's define

$$I_\epsilon = \int_{-\infty}^{\infty} dx \frac{(x - x_0) f(x)}{(x - x_0)^2 + \epsilon^2}. \tag{52}$$

Then $\lim_{\epsilon\to 0^+} I_\epsilon$ is defined as the principle value of I. This is denoted as

$$P\int_{-\infty}^{\infty} dx\ \frac{f(z)}{x-x_0}. \tag{53}$$

Now let's evaluate I_ϵ for small ϵ. We have

$$\begin{aligned}\frac{z-x_0}{(z-x_0)^2+\epsilon^2} &= \frac{z-x_0}{(z-x_0-i\epsilon)(z-x_0+i\epsilon)} \\ &= \frac{1}{z-x_0+i\epsilon}+\frac{i\epsilon}{(z-x_0-i\epsilon)(z-x_0+i\epsilon)}.\end{aligned} \tag{54}$$

Note that the first term has a pole at $x_0 - i\epsilon$ which is outside the contour we've established but the second term has a pole at $x_0+i\delta$. Then we can perform the contour integral associated with I_ϵ as

$$I_\epsilon = \oint_C dz \frac{f(z)}{z-x_0+i\epsilon} + 2\pi i f(x_0+i\epsilon)\frac{i\epsilon}{2i\epsilon}. \tag{55}$$

Then

$$P\int_{-\infty}^{\infty} dx\ \frac{f(z)}{x-x_0} = \lim_{\epsilon\to 0^+}\oint_C dz \frac{f(z)}{z-x_0+i\epsilon} + i\pi f(x_0) \tag{56}$$

An interesting exercise is to show that the limit $\epsilon \to 0^-$ gives the same overall result.

Finally, we define

$$P\oint_C dz \frac{f(z)}{z-x_0} \tag{57}$$

to be the result of displacing the pole slightly *inside* the contour.

6.1 Fun fact

Here's a fun fact. Suppose that $f(z)$ is holomorphic in the upper half-plane. Then

$$\begin{aligned}P\int_{-\infty}^{\infty} dx\ \frac{f(z)}{x-x_0} &= \lim_{\epsilon\to 0^+}\oint_C dz \frac{f(z)}{z-x_0+i\epsilon} + i\pi f(x_0) \\ &= i\pi f(x_0).\end{aligned} \tag{58}$$

Let's now write $f(z) = u(z) + iv(z)$. Then

$$u(x_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty} dx\ \frac{v(z)}{x-x_0} \tag{59}$$

$$v(x_0) = -\frac{1}{\pi}P\int_{-\infty}^{\infty} dx\ \frac{u(z)}{x-x_0} \tag{60}$$

7 Logarithms and branch points

We want to define the natural logarithm, $\ln z$ to be the inverse of e^z. In other words,

$$e^{\ln z} = z. \tag{61}$$

What can we say about the function $\ln z$? For one thing, $e^{i2\pi} = 1$ implies that $\ln z$ can only be defined up to $2\pi i$. That is, $\ln z$ is multiply-valued. That, in itself, is odd. That means that

$$\ln e^z = z + i2\pi n, \tag{62}$$

for some integer n.

Now let's look at $\partial e^{\ln z} = 1$. By the chain rule, $(\partial \ln z)\, e^{\ln z} = z(\partial \ln z) = 1$. Therefore $\partial \ln z = 1/z$. A nice way to think about $\ln z$ is

$$\ln z - \ln z_1 = \int_C \frac{dw}{w}, \tag{63}$$

where C is a contour that starts at z_1 and goes to z. We can see the multivaluedness by taking the unit circle as a contour, so that

$$\ln z - \ln z_1 = 2\pi i. \tag{64}$$

Now choose a contour starting on the real axis at point $z_1 = r$ going to $z = re^{i\theta}$. Then

$$\ln[re^{i\theta}] - \ln r = i\theta + 2\pi i N, \tag{65}$$

where N is the number of times the contour goes around the origin until it reaches z. Finally, we have

$$\ln z = \ln r + i\theta + 2\pi i N. \tag{66}$$

The origin is a **branch point**. If we insist (and eventually we won't have to) that $\ln z$ should be single valued, then $\ln z$ must have a line along which it is discontinuous. Such a line is called a **branch cut**.

Let's now be specific. We define a ray $z(r) = re^{i\theta_0}$ along which $\ln z$ will be discontinuous. We can then define $\ln z$ by integrating along a suitable contour. Let's start on the real axis at point r (or, if $\theta_0 = 0$, just above it real axis). Then

$$\begin{aligned} \ln z &= \ln r + \int_R \frac{dz}{z} \\ &= \ln r + i\theta,\ \theta < \theta_0. \end{aligned} \tag{67}$$

This is fine so long as the contour doesn't cross the branch cut. To get to a point on the other side of the branch cut, we integrate along the other direction,

$$\ln z = \ln r + \int_0^{-\theta} d\xi. \tag{68}$$

Working with branch cuts is not too difficult when we can deform our contours to avoid them. Notice that the discontinuity across the branch cut is always the same – $2\pi i$.

7.1 Other functions with branch points

Consider the function $z^{1/2}$. Integrating along a circular contour around the origin gives

$$\oint dz z^{1/2} = iR^{3/2} \int_0^{2\pi} d\theta e^{3i\theta/2} = -\frac{4}{3} R^{3/2}. \tag{69}$$

This represents the fact that the square root can be multiply valued. More specifically, note that $z^{1/2} = e^{(1/2)\ln z}$. Since $\ln z$ requires a branch cut across which it appears to have a discontinuity of $2\pi i$, $z^{1/2}$ has a discontinuity of $e^{i\pi} = -1$. Every time you circle the branch point, you pick up a factor of -1. If you traverse the branch point twice, however, $z^{1/2}$ returns to its original value.

With the branch cuts, square roots do not always follow the rules you know and love. For example, $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$ when we choose the branch cuts along the negative real axis. Why? Because the right-hand side is not defined on the negative real axis – both square roots have an issue – but the left-hand side is because zw is positive on the negative real axis.

7.2 An integral with a branch cut

Consider the integral

$$I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}. \tag{70}$$

We need to cook up an analytic function and a contour integral that gives the answer to this integral. The first problem we face is that there is no unique way to do this. For example, should $f(z) = \sqrt{z(1-z)}$, $f(z) = \sqrt{z}\sqrt{1-z}$ or $f(z) = z\sqrt{1-1/z}$? Well, a clue is seen in the integral itself. The first choice, $f(z) = \sqrt{z(1-z)}$ has a branch point at 0 and 1, as does $\sqrt{z}\sqrt{1-z}$.

One the other hand, the advantage of $f(z) = z\sqrt{1-1/z}$ is that it is composed of an analytic function, z, and a function with a branch point at $z = 1$. It is also well-defined along the real axis between 0 and 1, which is key for making sense of the integral. This makes the computation easier. Let's also choose the branch cut along the negative real axis. Then we have

$$I = \int_C dz \frac{1}{z\sqrt{1-1/z}}. \tag{71}$$

Let's look more carefully at what happens as $z = x + iy$. Then $1 - 1/z = 1 - 1/(x+iy) = 1 - (x-iy)/(x^2+y^2) = (x^2-x)/(x^2+y^2) + iy/(x^2+y^2)$. The real part of this expression is negative when x is between 0 and 1. Therefore, Therefore,

$$\lim_{y\to 0} f(x+iy) = \operatorname{sgn}(y)\left(i\frac{1}{x}\frac{1}{\sqrt{1-1/x}}\right). \tag{72}$$

But this is as we expect – the branch cut we choice induces a discontinuity in the integrand. Here is the cool thing – consider a rectangular contour that goes around the branch cut but stays a distance ϵ from it. This traverses the branch cut above and below it. Since we want to avoid going through the branch cut, we must contain the pole at $z = 0$ as well – therefore, we conclude

$$2i\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \oint_C dz f(z) = 2\pi i \tag{73}$$

where the last equality is from the residue theorem. Hence, $I = \pi$.