Physics 605: Integral transforms

Due: never

1 From Fourier series to transforms

Let's start with the Hilbert space of smooth functions periodic in the range $[-L, L)$ with inner product

$$\langle f|g\rangle = \int_{-L/2}^{L/2} dx\ \bar{f}(x)g(x). \tag{1}$$

Since the operator $i\partial_x$ is self-adjoint, its eigenfunctions form a basis for the Hilbert space. These induce a basis of orthonormal functions, $\mathcal{B} = \{e^{2i\pi mx/L}/\sqrt{L}\}$. This means, in particular, that

$$f(x) = \sum_{m=-\infty}^{\infty} \frac{c_m}{\sqrt{L}} e^{2\pi imx/L} \tag{2}$$

and

$$c_m = \int_{-L/2}^{L/2} dx \frac{1}{\sqrt{L}} e^{-2\pi imx/L} f(x). \tag{3}$$

Our intention is to try to take the limit that $L \to \infty$. Before we do this, let's define a "function" $F(x)$ such that $F(2\pi m/L) = c_m/\sqrt{L}$. Then

$$f(x) = \frac{1}{L} \sum_{m=-\infty}^{\infty} F(2\pi m/L) e^{2\pi imx/L} \tag{4}$$

$$F(2\pi m/L) = \int_{-L/2}^{L/2} dx e^{-2\pi imx/L} f(x). \tag{5}$$

Let's define $k_m = (2\pi/L)m$. Then $\Delta k = 2\pi/L$ is the spacing between adjacent points.

Let's finally write

$$f(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \Delta k\ F(2\pi m/L) e^{2\pi imx/L}. \tag{6}$$

Now we can take the limit that $L \to \infty$. In that limit, $\Delta k \to 0$. We see that Eq. (6) is the definition of a Riemann integral for the function $F(k)$. Having rewritten things so conveniently, taking the limit that $L \to \infty$ gives

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} dx e^{-ikx} f(x) && (7) \\ f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k). && (8) \end{aligned}$$

These relations define the **Fourier transform**. Note that $F(k)$ has units of $f(x)$ times length and k has units of $1/L$.

Let's check, explicitly, that these are inverses of each other. Then

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx'\ e^{ik(x-x')} f(x'). \tag{9}$$

If we can switch the order of the integrals then

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x'). \tag{10}$$

This calculation also tells us something about what functions we are "allowed" to Fourier transform. These are functions for which the integrals converge sufficiently nicely that the order of integrals can be switched in the above

Sometimes we denote the Fourier transform as $\mathcal{F}[f](k)$ and its inverse as $\mathcal{F}^{-1}[F](x)$.

1.1 Properties of the Fourier transform

Here are some of the important properties of the Fourier transform.

Theorem: $\mathcal{F}[\partial_x f] = ik\mathcal{F}[f](k)$.

proof:

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{-ikx} \partial_x f(x) &= -\int_{-\infty}^{\infty} dx \left[\partial_x e^{-ikx}\right] f(x) \\ &= ik \int_{-\infty}^{\infty} dx e^{-ikx} f(x). && (11) \end{aligned}$$

Indeed, we can integrate-by-parts *ad nauseum* to prove

$$\mathcal{F}[\partial_x^n f] = (ik)^n \mathcal{F}[f](k). \tag{12}$$

Theorem: (Convolution) $\mathcal{F}[f(x)g(x)](k) = \int \frac{dq}{2\pi} \mathcal{F}[f](q)\mathcal{F}[g](k-q)$.

proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty} dx e^{-ikx} f(x) g(x) &= \int_{-\infty}^{\infty} dx\ e^{-ikx} g(x) \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iqx} \mathcal{F}[f](q) \\
&= \int \frac{dq}{2\pi} F(q) \int_{-\infty}^{\infty} dx\ e^{-i(k-q)x} g(x) \\
&= \int \frac{dq}{2\pi} \mathcal{F}[f](q) \mathcal{F}[g](k-q).
\end{aligned}
$$

Theorem: (Convolution 2) $\mathcal{F}^{-1}[F(q)G(q)](x) = \int dx' \mathcal{F}^{-1}[F](x') \mathcal{F}^{-1}[G](x - x')$.

proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k) g(k) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ikx} G(k) \int_{-\infty}^{\infty} dx' e^{-ikx'} \mathcal{F}^{-1}[F](x') \\
&= \int dx' G(k) \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ik(x-x')} f(x') \\
&= \int dx' \mathcal{F}^{-1}[G](x-x') \mathcal{F}^{-1}[F](x').
\end{aligned}
$$

2 Green Functions from Fourier Transforms

Consider the equation for the forced, damped simple harmonic oscillator,

$$
\partial_t^2 h(t) + \gamma \partial_t h(t) + \omega_0^2 h(t) = f(t). \tag{13}
$$

Suppose we Fourier transform both sides of this equation. By convention, instead of using the variable k, we'll use ω since this is a Fourier transform in time. We will keep the *same* signs for our Fourier transforms as previously defined, however. On the right-hand side, $\mathcal{F}[f](\omega) \equiv F(\omega)$. On the left-hand side,

$$
\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) H(\omega) = F(\omega), \tag{14}
$$

where $\mathcal{F}[h](\omega) = H(\omega)$. Our differential equation therefore becomes algebraic in "Fourier space." Then we have

$$
H(\omega) = \frac{F(\omega)}{-\omega^2 + i\gamma\omega + \omega_0^2}. \tag{15}
$$

If we Fourier transform back, the convolution theorem tells us that

$$
h(t) = \int dt'\ G(t-t') f(t'), \tag{16}
$$

where

$$G(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{-\omega^2 + i\gamma\omega + \omega_0^2} \tag{17}$$

Eq. (16) is the equation defining a Green function while Eq. (17) gives us an integral expression for precisely that Green function.

To really drive this home, let's see what happens when we act on Eq. (17) with $\partial_t^2 + \gamma\partial_t + \omega_0^2$. Then we obtain

$$\begin{aligned}\left(\partial_t^2 + \gamma\partial_t + \omega_0^2\right) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{-\omega^2 + i\gamma\omega + \omega_0^2} &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) e^{i\omega(t-t')}}{-\omega^2 + i\gamma\omega + \omega_0^2} \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t-t')} = \delta(t-t').\end{aligned}$$

This is precisely what we defined our Green function to be previously.

Now we can solve for the Green function by directly solving the equation

$$\partial_t^2 G(t) + \gamma\partial_t G(t) + \omega_0^2 G(t) = \delta(t-t'). \tag{18}$$

However, we may also obtain the Green function directly by doing the inverse Fourier transform. To do the integral, we turn it into an integral on the complex plane on a contour lying on the real axis. The integrand has two poles,

$$\omega = \frac{i}{2}\gamma \pm \sqrt{\omega_0^2 - \gamma^2/4}. \tag{19}$$

If $\omega_0^2 > 0$ and $\gamma > 0$, the imaginary part of these poles is always positive. The contour integral we want to do is

$$\begin{aligned}G(t-t') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{-\omega^2 + i\gamma\omega + \omega_0^2} = \lim_{R\to\infty} \left\{ \int_C \frac{dw}{2\pi} \frac{e^{i(t-t')w}}{-\omega^2 + i\gamma\omega + \omega_0^2} \right. \\ &\quad \left. - \int d\theta i R e^{i\theta} \frac{e^{i(t-t')Re^{i\theta}}}{-R^2 e^{2i\theta} + i\gamma R e^{i\theta} + \omega_0^2} \right\}.\end{aligned} \tag{20}$$

If $t - t' < 0$, we close the contour below the real axis. Thus, we contain no poles! If, on the other hand, $t - t' > 0$, we close the contour above the real axis and the contour contains two poles. The integral is then

$$\begin{aligned}G(t-t') &= -e^{-(t-t')\gamma/2} \left[\frac{e^{-i(t-t')\sqrt{\omega_0^2-\gamma^2/4}}}{\gamma - i\sqrt{4\omega_0^2 - \gamma^2}} + \frac{e^{i(t-t')\sqrt{\omega_0^2-\gamma^2/4}}}{\gamma + i\sqrt{4\omega_0^2 - \gamma^2}} \right] \Theta(t-t') \\ &= -\Theta(t-t') \frac{e^{-(t-t')\gamma/2}}{2\omega_0^2} \left[\gamma \cos\left((t-t')\sqrt{\omega_0^2 - \gamma^2/4} \right) \right. \\ &\quad \left. + \sqrt{4\omega_0^2 - \gamma^2} \sin\left((t-t')\sqrt{\omega_0^2 - \gamma^2/4} \right) \right]\end{aligned} \tag{21}$$

2.1 Green function for the wave equation in 3D

We will now use these techniques to study the Green function for the wave equation in three dimension. In particular, we are solving the equation

$$\left(-\nabla^2 + \frac{1}{c^2}\partial_t^2\right) g(\mathbf{x}, t; \mathbf{x}', t') = 4\pi\delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{22}$$

There are four variables, so we proceed to Fourier transform as

$$G(\mathbf{k}, \omega) = \int d^3x dt \; e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t} g(\mathbf{x}, t). \tag{23}$$

Notice the change in sign for ω; this is standard convention for the wave equation that arises from considerations of special relativity. It changes nothing other than the sign of ω.

In Fourier space, $\nabla h \to i\mathbf{k}h$ and

$$\delta^3(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{24}$$

$$\delta(t) = \int \frac{dt}{2\pi} e^{-i\omega t}. \tag{25}$$

Therefore, the wave equation, in Fourier space, becomes

$$\left(\mathbf{k}^2 - \omega^2/c^2\right) G(\mathbf{k}, \omega) = 4\pi e^{-i\mathbf{k}\cdot\mathbf{x}' + i\omega t'}. \tag{26}$$

The solution in real space is found by Fourier transforming back as

$$g(\mathbf{x}, t; \mathbf{x}', t') = \frac{2}{(2\pi)^3} \int d^3k d\omega \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}}{\mathbf{k}^2 - \omega^2/c^2}. \tag{27}$$

One thing we see immediately is that the Green function is actually a function of $\mathbf{x} - \mathbf{x}'$ and $t - t'$. This turns out to be a consequence of the translation symmetry of the equation.

The first thing we want to do is write this in spherical coordinates. If we orient our coordinate system so that the vector $\mathbf{x} - \mathbf{x}'$ points along the north pole, we have

$$\begin{aligned} g(\mathbf{x} - \mathbf{x}', t - t') &= \frac{2c^2}{(2\pi)^3} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk k^2 \int_{-1}^{1} d(\cos\theta) d\varphi \frac{e^{ik|\mathbf{x}-\mathbf{x}'|\cos\theta} e^{-i\omega(t-t')}}{c^2k^2 - \omega^2} \\ &= \frac{c^2}{\pi^2|\mathbf{x} - \mathbf{x}'|} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk \frac{k}{c^2k^2 - \omega^2} \sin\left(k|\mathbf{x} - \mathbf{x}'|\right) e^{-i\omega(t-t')} \\ &= \frac{c^2}{2\pi^2 i|\mathbf{x} - \mathbf{x}'|} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \frac{ke^{-i\omega(t-t')}}{c^2k^2 - \omega^2} e^{ik|\mathbf{x}-\mathbf{x}'|} \end{aligned} \tag{28}$$

The form of this integral suggests turning it into a contour integral. The difficulty is that the contour would be over the real axis but the poles of k are $\pm\omega/c$. The only solution to this problem is to deform the poles off the real axis slightly and do the integral as a limit. The result is that we have four ways to do this: (1) both poles can have slightly positive imaginary parts, (2) negative imaginary parts, (3) $+\omega/c$ could move up while $-\omega/c$ moves down, or (4) $-\omega/c$ moves up while ω/c moves down. How do we decide? Let's choose one and see what happens; then consider what would have happened if we had made a different choice.

Suppose we move both poles upward to have a positive imaginary part. Then, since $|\mathbf{x}-\mathbf{x}'| > 0$, we turn the integral over k into a contour integral along a semicircle in the upper half-plane. Then when the radius of that semicircle becomes infinite, the contribution from the arc vanishes (prove this!). Finally, we have

$$\begin{aligned} g(\mathbf{x}-\mathbf{x}', t-t') &= \frac{1}{2\pi|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t'+|\mathbf{x}-\mathbf{x}'|/c)} \qquad (29) \\ &\quad + \frac{1}{2\pi|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t'-|\mathbf{x}-\mathbf{x}'|/c)} \\ &= \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} + \frac{\delta(t-t'+|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \qquad (30) \end{aligned}$$

The first delta function is zero unless $t = t' + |\mathbf{x}'-\mathbf{x}|/c$ while the second is zero except for $t = t' - |\mathbf{x}-\mathbf{x}'|/c$.

3 Laplace transform

The Laplace transform is, sort of, a kind of one-sided Fourier transform. To motivate it, consider the following equation:

$$(\partial_t + m)\, u(t) = 0 \qquad (31)$$

with initial condition $u(0) = u_0$ and $m > 0$. We could incorporate this initial condition with a slight modification of our equation to

$$(\partial_t + m)\, u(t) = u_0\delta(t). \qquad (32)$$

Integrating both sides by t around $t = 0$ and setting $u(0^-) = 0$ gives us precisely what we want. In Fourier space, we have

$$(i\omega + m)U(\omega) = u_0 \qquad (33)$$

so that

$$u(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{u_0}{\omega - im} e^{i\omega t} = \Theta(t) u_0 e^{-mt}. \tag{34}$$

If $m < 0$, however, we obtain

$$u(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{u_0}{\omega + i|m|} e^{i\omega t} = \Theta(-t) u_0 e^{|m|t}. \tag{35}$$

The Laplace transform allows us to solve this as an initial-value problem directly. It is defined as follows:

$$U(w) = \int_{0^-}^{\infty} dt e^{-wt} u(t) \tag{36}$$

$$u(t) = \int_C \frac{dw}{2\pi i} e^{wt} U(w). \tag{37}$$

We will have to choose the contour C very carefully. Let's start by asking whether these are truly inverses of each other. First,

$$\begin{aligned} u(t) &= \int_C \frac{dw}{2\pi} e^{wt} \left[\int_{0^-}^{\infty} dt' e^{-wt'} u(t') \right] \\ &= \int_{0^-}^{\infty} dt' \int_C \frac{dw}{2\pi i} e^{w(t-t')} u(t'). \end{aligned} \tag{38}$$

To make this work, we can choose a contour $w(y) = \alpha + iy$ integrated from negative to positive infinity (we'll come back to α in a second). In that case,

$$\int_C \frac{dw}{2\pi i} e^{w(t-t')} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\ e^{\alpha(t-t')} e^{iy(t-t')} = \delta(t-t'). \tag{39}$$

Therefore, $u(t) = \int_{0^-}^{\infty} dt' \delta(t-t') u(t')$.

Now let's look at

$$\begin{aligned} U(w) &= \int_{0^-}^{\infty} dt e^{-wt} \left[\int_C \frac{dz}{2\pi i} e^{zt} U(z) \right] \\ &= \int_C \frac{dz}{2\pi i} \frac{U(z)}{z - w}. \end{aligned} \tag{40}$$

This works so long as we choose a contour to the right of all the poles of $U(w)$, close our infinite contour to the left, and have $\mathrm{Re} z > \mathrm{Re} w$.

One of the advantages of working with the Laplace transform is

$$\int_{0^-}^{\infty} dt e^{-wt} \partial_t f(t) = w \int_{0^-}^{\infty} dt e^{-wt} f(t) - f(0). \tag{41}$$

So let's look at the Laplace transform of our equation: $(\partial_t + m)u(t) = 0$. After a Laplace transform, this becomes $(w+m)U(w) - u(0) = 0$. Therefore,

$$U(w) = \frac{u(0)}{w+m} \tag{42}$$

so

$$u(t) = u(0) \int_C \frac{dw}{2\pi i} \frac{e^{wt}}{w+m}. \tag{43}$$

Since the contour is to the right of the pole at $w = -m$ and the contour closes to the left, we find $u(t) = u(0)e^{-mt}$ as we expect.

4 The Hankel transform

There are many other integral transforms available. The only other one we want to mention is the Hankel transform. We already know that, in cylindrical coordinates, we can solve Laplace's equation with a series of Bessel functions. The Hankel transform turns this series into an integral transform, much as the Fourier transform is the continuum limit of a Fourier series. In particular,

$$H_\nu(k) = \int_0^\infty dr\ rJ_\nu(kr)h(r),\ \nu > -1/2 \tag{44}$$

$$h(r) = \int_0^\infty dk\ k\ J_\nu(kr)H_\nu(k). \tag{45}$$

We can relate this transform to the Fourier transform in 2D. We write

$$H(k\hat{\mathbf{k}}) = \int drrd\theta e^{ikr\cos\theta}h(r,\theta). \tag{46}$$

Now expand $h(r,\theta) = \sum_{m=-\infty}^{\infty} h_m(r)e^{im\theta}$ so that

$$H(k\hat{\mathbf{k}}) = \sum_{m=-\infty}^{\infty} \int drrd\theta\ e^{ikr\cos\theta + im\theta}h_m(r). \tag{47}$$

Then

$$H(k) = \sum_{m=-\infty}^{\infty} \int_0^\infty drrd\theta J_m(kr)h_{-m}(r), \tag{48}$$

using the integral representation of the Bessel function

$$\int_0^{2\pi} d\theta e^{ikr\cos\theta - im\theta} = J_m(kr). \tag{49}$$

The Hankel transform is what we get when we select the integral for one particular m.